Moscow Mathematical Olympiad 1956

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– tour 1

320 Prove that there are no four points A, B, C, D on a plane such that all triangles $\triangle\ ABC, \triangle\ BCD, \triangle\ CDA, \triangle\ DAB$ are acute ones.

321 Find all two-digit numbers x the sum of whose digits is the same as that of $2x$, $3x$, ... , $9x$.

322 A closed self-intersecting broken line intersects each of its segments once. Prove that the number of its segments is even.

323 a) Find all integers that can divide both the numerator and denominator of the ratio $\frac{5m+6}{8m+7}$ for an integer m.
b) Let a, b, c, d, m be integers. Prove that if the numerator and denominator of the ratio $\frac{am+b}{cm+d}$ are both divisible by k, then so is $ad - bc$.

324 a) What is the least number of points that can be chosen on a circle of length 1956, so that for each of these points there is exactly one chosen point at distance 1, and exactly one chosen point at distance 2 (distances are measured along the circle)?
b) On a circle of length 15 there are selected n points such that for each of them there is exactly one selected point at distance 1 from it, and exactly one is selected point at distance 2 from it. (All distances are measured along the circle.) Prove that n is divisible by 10.

325 On sides AB and CB of $\triangle\ ABC$ there are drawn equal segments, AD and CE, respectively, of arbitrary length (but shorter than min(AB, BC)). Find the locus of midpoints of all possible segments DE.

326 a) In the decimal expression of a positive number, a, all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a with accuracy to 0.01 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b. What numbers b can be thus obtained? Write all their possible values.
b) same as (a) but with accuracy to 0.001
c) same as (a) but with accuracy to 0.0001

327 On an infinite sheet of graph paper a table is drawn so that in each square of the table stands a number equal to the arithmetic mean of the four adjacent numbers. Out of the table a piece is cut along the lines of the graph paper. Prove that the largest number on the piece always occurs at an edge, where $x = \frac{1}{4}(a+b+c+d)$.

328 In a convex quadrilateral $ABCD$, consider quadrilateral $KLMN$ formed by the centers of mass of triangles ABC, BCD, DBA, CDA. Prove that the straight lines connecting the midpoints of the opposite sides of quadrilateral $ABCD$ meet at the same point as the straight lines connecting the midpoints of the opposite sides of $KLMN$.

329 Consider positive numbers h, s_1, s_2, and a spatial triangle $\triangle ABC$. How many ways are there to select a point D so that the height of tetrahedron $ABCD$ drawn from D equals h, and the areas of faces ACD and BCD equal s_1 and s_2, respectively?

330 A square of side a is inscribed in a triangle so that two of the square's vertices lie on the base, and the other two lie on the sides of the triangle. Prove that if r is the radius of the circle inscribed in the triangle, then $r\sqrt{2} < a < 2r$.

331 Given a closed broken line $A_1A_2A_3...A_n$ in space and a plane intersecting all its segments, A_1A_2 at B_1, A_2A_3 at $B_2, ..., A_nA_1$ at Bn, prove that

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot ... \cdot \frac{A_nB_n}{B_nA_1} = 1$$

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332 Prove that the system of equations $\begin{cases} x_1 - x_2 = a \\ x_3 - x_4 = b \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$

has at least one solution in positive numbers ($x_1, x_2, x_3, x_4 > 0$) if and only if $|a| + |b| < 1$.

– tour 2

333 Let O be the center of the circle circumscribed around $\triangle ABC$, let A_1, B_1, C_1 be symmetric to O through respective sides of $\triangle ABC$. Prove that all heights of $\triangle A_1B_1C_1$ pass through O, and all heights of $\triangle ABC$ pass through the center of the circle circumscribed around $\triangle A_1B_1C_1$.

334 a) Points $A_1, A_2, A_3, A_4, A_5, A_6$ divide a circle of radius 1 into six equal arcs. Ray ℓ_1 from A_1 connects A_1 with A_2, ray ℓ_2 from A_2 connects A_2 with A_3, and so on, ray ℓ_6 from A_6 connects A_6 with A_1. From a point B_1 on ℓ_1 the perpendicular is drawn on ℓ_6, from the foot of this perpendicular another perpendicular is drawn on ℓ_5, and so on. Let the foot of the 6-th perpendicular coincide with B_1. Find the length of segment A_1B_1.

b) Find points $B_1, B_2, ..., B_n$ on the extensions of sides $A_1A_2, A_2A_3, ..., A_nA_1$ of a regular n-gon $A_1A_2...A_n$ such that $B_1B_2 \perp A_1A_2$, $B_2B_3 \perp A_2A_3$,..., $B_nB_1 \perp A_nA_1$.

335 a) 100 numbers (some positive, some negative) are written in a row. All of the following three types of numbers are underlined: 1) every positive number, 2) every number whose sum with the number following it is positive, 3) every number whose sum with the two numbers following it is positive.
Can the sum of all underlined numbers be
(i) negative?
(ii) equal to zero?

b) n numbers (some positive and some negative) are written in a row. Each positive number and each number whose sum with several of the numbers following it is positive is underlined. Prove that the sum of all underlined numbers is positive.

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336 64 non-negative numbers whose sum equals 1956 are arranged in a square table, eight numbers in each row and each column. The sum of the numbers on the two longest diagonals is equal to 112. The numbers situated symmetrically with respect to any of the longest diagonals are equal.
(a) Prove that the sum of numbers in any column is less than 1035.
(b) Prove that the sum of numbers in any row is less than 518.

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337 * Assume that the number of a tree's leaves is a multiple of 15. Neglecting the shade of the trunk and branches prove that one can rip off the tree $7/15$ of its leaves so that not less than $8/15$ of its shade remains.

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338 * A shipment of 13.5 tons is packed in a number of weightless containers. Each loaded container weighs not more than 350 kg. Prove that 11 trucks each of which is capable of carrying $\cdot\, 1.5$ ton can carry this load.

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339 Find the union of all projections of a given line segment AB to all lines passing through a given point O.

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340 a) * In a rectangle of area 5 sq. units, 9 rectangles of area 1 are arranged. Prove that the area of the overlap of some two of these rectangles is $\geq 1/9$
b) In a rectangle of area 5 sq. units, lie 9 arbitrary polygons each of area 1. Prove that the area of the overlap of some two of these rectangles is $\geq 1/9$

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341 1956 points are chosen in a cube with edge 13.
Is it possible to fit inside the cube a cube with edge 1 that would not contain any of the selected points?

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342 Given three numbers x, y, z denote the absolute values of the differences of each pair by x_1, y_1, z_1. From x_1, y_1, z_1 form in the same fashion the numbers x_2, y_2, z_2, etc. It is known that $x_n = x, y_n = y, z_n = z$ for some n. Find y and z if $x = 1$.

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343 A quadrilateral is circumscribed around a circle. Prove that the straight lines connecting neighboring tangent points either meet on the extension of a diagonal of the quadrilateral or are parallel to it.

344 * Let A, B, C be three nodes of a graph paper. Prove that if $\triangle ABC$ is an acute one, then there is at least one more node either inside $\triangle ABC$ or on one of its sides.

345 * Prove that if the trihedral angles at each of the vertices of a triangular pyramid are formed by the identical planar angles, then all faces of this pyramid are equal.