National Matematical Olympiad 2015

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–	2nd Round
1	In an acute-angled triangle $\triangle ABC$, D is the point on BC such that AD bisects $\angle BAC$, E and F are the feet of the perpendiculars from D onto AB and AC respectively. The segments BF and CE intersect at K. Prove that AK is perpendicular to BC.
2	A boy lives in a small island in which there are three roads at every junction. He starts from his home and walks along the roads. At each junction he would choose to turn to the road on his right or left alternatively, i.e., his choices would be . . ., left, right, left,... Prove that he will eventually return to his home.
3	Find all functions $f:\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, such that $f(x)f(yf(x)-1) = x^2f(y)-f(x) \quad \forall x,y \in \mathbb{R}$
4	Let $f_0, f_1, ...$ be the Fibonacci sequence: $f_0 = f_1 = 1, f_n = f_{n-1} + f_{n-2}$ if $n \geq 2$. Determine all possible positive integers n so that there is a positive integer a such that $f_n \leq a \leq f_{n+1}$ and that $a(\frac{1}{f_1}+\frac{1}{f_1f_2}+\cdots+\frac{1}{f_1f_2...f_n})$ is an integer.
5	Let n ¿ 3 be a given integer. Find the largest integer d (in terms of n) such that for any set S of n integers, there are four distinct (but not necessarily disjoint) nonempty subsets, the sum of the elements of each of which is divisible by d.