Moscow Mathematical Olympiad 1957

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by parmenides51

– tour 1

346 Find all isosceles trapezoids that are divided into 2 isosceles triangles by a diagonal.

347 a) Let $ax^3 + bx^2 + cx + d$ be divisible by 5 for given positive integers a, b, c, d and any integer x. Prove that a, b, c and d are all divisible by 5.
b) Let $ax^4 + bx^3 + cx^2 + dx + e$ be divisible by 7 for given positive integers a, b, c, d, e and all integers x.
Prove that a, b, c, d and e are all divisible by 7.

348 A snail crawls over a table at a constant speed. Every 15 minutes it turns by 90^o, and in between these turns it crawls along a straight line. Prove that it can return to the starting point only in an integer number of hours.

349 For any column and any row in a rectangular numerical table, the product of the sum of the numbers in a column by the sum of the numbers in a row is equal to the number at the intersection of the column and the row. Prove that either the sum of all the numbers in the table is equal to 1 or all the numbers are equal to 0.

350 The distance between towns A and B is 999 km.
At every kilometer of the road that connects A and B a sign shows the distances to A and B as follows: 0-999 , 1-998 , 2-997 , ... , 998-1 , 999-0
How many signs are there, with both distances written with the help of only two distinct digits?

351 Given two concentric circles and a pair of parallel lines. Find the locus of the fourth vertices of all rectangles with three vertices on the concentric circles, two vertices on one circle and the third on the other and with sides parallel to the given lines.

352 Of all parallelograms of a given area find the one with the shortest possible longer diagonal.

353 Solve the equation $x^3 - [x] = 3$.

354 In a quadrilateral $ABCD$ points M and N are the midpoints of the diagonals AC and BD, respectively. The line through M and N meets AB and CD at M' and N', respectively. Prove that if $MM' = NN'$, then $AD//BC$.

355 a) A student takes a subway to an Olympiad, pays one ruble and gets his change. Prove that if he takes a tram (street car) on his way home, he will have enough coins to pay the fare without change.
b) A student is going to a club. (S)he takes a tram, pays one ruble and gets the change. Prove that on the way back by a tram (s)he will be able to pay the fare without any need to change.

Note: In 1957, the price of a subway ticket was 50 kopeks, that of a tram ticket 30 kopeks, the denominations of the coins were $1, 2, 3, 5, 10, 15$, and 20 kopeks. (1 rouble = 100 kopeks.)

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356 A planar polygon $A_1A_2A_3...A_{n-1}A_n$ ($n > 4$) is made of rigid rods that are connected by hinges. Is it possible to bend the polygon (at hinges only!) into a triangle?

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357 For which integer n is $N = 20^n + 16^n - 3^n - 1$ divisible by 323?

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358 The segments of a closed broken line in space are of equal length, and each three consecutive segments are mutually perpendicular. Prove that the number of segments is divisible by 6.

– tour 2

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359 Straight lines OA and OB are perpendicular. Find the locus of endpoints M of all broken lines OM of length ℓ which intersect each line parallel to OA or OB at not more than one point.

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360 (a) A radio lamp has a 7-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 7 holes. Is it possible to number the contacts and the holes so that for any insertion at least one contact would match the hole with the same number?

(b) A radio lamp has a 20-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 20 holes. Let the contacts in the plug and the socket be already numbered. Is it always possible to insert the plug so that none of the contacts matches its socket?

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361 The lengths, a and b, of two sides of a triangle are known.

(a) What length should the third side be, in order for the largest angle of the triangle to be of the least possible value?
(b) What length should the third side be in order for the smallest angle of the triangle to be of the greatest possible value?

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362 (a) A circle is inscribed in a triangle. The tangent points are the vertices of a second triangle in which another circle is inscribed. Its tangency points are the vertices of a third triangle. The angles of this triangle are identical to those of the first triangle. Find these angles.

(b) A circle is inscribed in a scalene triangle. The tangent points are vertices of another triangle, in which a circle is inscribed whose tangent points are vertices of a third triangle, in which a third circle is inscribed, etc. Prove that the resulting sequence does not contain a pair of similar triangles.

363 Eight consecutive numbers are chosen from the Fibonacci sequence $1, 2, 3, 5, 8, 13, 21, ...$. Prove that the sequence does not contain the sum of chosen numbers.

364 (a) Prove that the number of all digits in the sequence $1, 2, 3, ..., 10^8$ is equal to the number of all zeros in the sequence $1, 2, 3, ..., 10^9$.

(b) Prove that the number of all digits in the sequence $1, 2, 3, ..., 10^k$ is equal to the number of all zeros in the sequence $1, 2, 3, ..., 10^{k+1}$.

365 (a) Given a point O inside an equilateral triangle $\triangle ABC$. Line OG connects O with the center of mass G of the triangle and intersects the sides of the triangle, or their extensions, at points A', B', C' . Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} = 3.$$

(b) Point G is the center of the sphere inscribed in a regular tetrahedron $ABCD$. Straight line OG connecting G with a point O inside the tetrahedron intersects the faces at points A', B', C', D'. Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} + \frac{D'O}{D'G} = 4.$$

366 Solve the system:

$$\begin{cases} \dfrac{2x_1^2}{1+x_1^2} = x_2 \\ \dfrac{2x_2^2}{1+x_2^2} = x_3 \\ \dfrac{2x_3^2}{1+x_3^2} = x_1 \end{cases}$$

367 Two rectangles on a plane intersect at eight points. Consider every other intersection point, they are connected with line segments, these segments form a quadrilateral. Prove that the area of this quadrilateral does not vary under translations of one of the rectangles.

368 Find all real solutions of the system :
(a)

$$\begin{cases} 1 - x_1^2 = x_2 \\ 1 - x_2^2 = x_3 \\ ... \\ 1 - x_{98}^2 = x_{99} \\ 1 - x_{99}^2 = x_1 \end{cases}$$

(b)*

$$\begin{cases} 1-x_1^2=x_2 \\ 1-x_2^2=x_3 \\ ... \\ 1-x_{98}^2=x_n \\ 1-x_n^2=x_1 \end{cases}$$

369 Represent 1957 as the sum of 12 positive integer summands $a_1, a_2, ..., a_{12}$ for which the number $a_1! \cdot a_2! \cdot a_3! \cdot ... \cdot a_{12}!$ is minimal.

370 * Three equal circles are tangent to each other externally and to the fourth circle internally. Tangent lines are drawn to the circles from an arbitrary point on the fourth circle. Prove that the sum of the lengths of two tangent lines equals the length of the third tangent.

371 Given quadrilateral $ABCD$ and the directions of its sides. Inscribe a rectangle in $ABCD$.

372 Given n integers $a_1 = 1, a_2, ..., a_n$ such that $a_i \le a_{i+1} \le 2a_i$ $(i = 1, 2, 3, ..., n-1)$ and whose sum is even. Find whether it is possible to divide them into two groups so that the sum of numbers in one group is equal to the sum of numbers in the other group.