Dutch Mathematical Olympiad 2005

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by Arne

1 In how many ways can one choose distinct numbers a and b from 1, 2, 3, ..., 2005 such that a + b is a multiple of 5?

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2 Let $P_1P_2P_3 \ldots P_{12}$ be a regular dodecagon. Show that

$$|P_1P_2|^2 + |P_1P_4|^2 + |P_1P_6|^2 + |P_1P_8|^2 + |P_1P_{10}|^2 + |P_1P_{12}|^2$$

is equal to

$$|P_1P_3|^2 + |P_1P_5|^2 + |P_1P_7|^2 + |P_1P_9|^2 + |P_1P_{11}|^2 .$$

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3 Let a_1, a_2, a_3, a_4, a_5 be distinct real numbers. Consider all sums of the form $a_i + a_j$ where $i, j \in \{1, 2, 3, 4, 5\}$ and $i \neq j$. Let m be the number of distinct numbers among these sums. What is the smallest possible value of m?

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4 Let $ABCD$ be a quadrilateral with $AB \parallel CD$, $AB > CD$. Prove that the line passing through $AC \cap BD$ and $AD \cap BC$ passes through the midpoints of AB and CD.

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5 Consider an array of numbers of size 8×8. Each of the numbers in the array equals 1 or -1. "Doing a move" means that you pick any number in the array and you change the sign of all numbers which are in the same row or column as the number you picked. (This includes changing the sign of the "chosen" number itself.) Prove that one can transform any given array into an array containing numbers +1 only by performing this kind of moves repeatedly.
