Dutch IMO Team Selection Test 2014

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–	Day 1
1	Determine all pairs (a, b) of positive integers satisfying $$a^2 + b \mid a^2b + a \quad \text{and} \quad b^2 - a \mid ab^2 + b.$$
2	Let $\triangle ABC$ be a triangle. Let M be the midpoint of BC and let D be a point on the interior of side AB. The intersection of AM and CD is called E. Suppose that $\lvert AD\rvert = \lvert DE\rvert$. Prove that $\lvert AB\rvert = \lvert CE\rvert$.
3	Let a, b and c be rational numbers for which $a + bc$, $b + ac$ and $a + b$ are all non-zero and for which we have $$\frac{1}{a+bc} + \frac{1}{b+ac} = \frac{1}{a+b}.$$ Prove that $\sqrt{(c-3)(c+1)}$ is rational.
4	Let $\triangle ABC$ be a triangle with $\lvert AC\rvert = 2\lvert AB\rvert$ and let O be its circumcenter. Let D be the intersection of the bisector of $\angle A$ with BC. Let E be the orthogonal projection of O to AD and let $F \neq D$ be the point on AD satisfying $\lvert CD\rvert = \lvert CF\rvert$. Prove that $\angle EBF = \angle ECF$.
5	On each of the 2014^2 squares of a 2014×2014-board a light bulb is put. Light bulbs can be either on or off. In the starting situation a number of the light bulbs is on. A move consists of choosing a row or column in which at least 1007 light bulbs are on and changing the state of all 2014 light bulbs in this row or column (from on to off or from off to on). Find the smallest non-negative integer k such that from each starting situation there is a finite sequence of moves to a situation in which at most k light bulbs are on.
–	Day 2
1	Let $f : \mathbb{Z}_{>0} \to \mathbb{R}$ be a function such that for all $n > 1$ there is a prime divisor p of n such that $$f(n) = f\left(\frac{n}{p}\right) - f(p).$$ Furthermore, it is given that $f(2^{2014}) + f(3^{2015}) + f(5^{2016}) = 2013$. Determine $f(2014^2) + f(2015^3) + f(2016^5)$.

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2 The sets A and B are subsets of the positive integers. The sum of any two distinct elements of A is an element of B. The quotient of any two distinct elements of B (where we divide the largest by the smallest of the two) is an element of A. Determine the maximum number of elements in $A \cup B$.

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3 Let H be the orthocentre of an acute triangle ABC. The line through A perpendicular to AC and the line through B perpendicular to BC intersect in D. The circle with centre C through H intersects the circumcircle of triangle ABC in the points E and F. Prove that $|DE| = |DF| = |AB|$.

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4 Determine all pairs (p, q) of primes for which $p^{q+1} + q^{p+1}$ is a perfect square.

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5 Let $P(x)$ be a polynomial of degree $n \leq 10$ with integral coefficients such that for every $k \in \{1, 2, \ldots, 10\}$ there is an integer m with $P(m) = k$. Furthermore, it is given that $|P(10) - P(0)| < 1000$. Prove that for every integer k there is an integer m such that $P(m) = k$.

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