Saudi Arabia IMO Team Selection Test 2014

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–	Day I
1	Tarik and Sultan are playing the following game. Tarik thinks of a number that is greater than 100. Then Sultan is telling a number greater than 1. If Tarik's number is divisible by Sultan's number, Sultan wins, otherwise Tarik subtracts Sultan's number from his number and Sultan tells his next number. Sultan is forbidden to repeat his numbers. If Tarik's number becomes negative, Sultan loses. Does Sultan have a winning strategy?
2	Define a *domino* to be an ordered pair of *distinct* positive integers. A *proper sequence* of dominoes is a list of distinct dominoes in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not *both* appear for any i and j. Let D_n be the set of all dominoes whose coordinates are no larger than n. Find the length of the longest proper sequence of dominoes that can be formed using the dominoes of D_n.
3	Let ABC be a triangle and let P be a point on BC. Points M and N lie on AB and AC, respectively such that MN is not parallel to BC and $AMPN$ is a parallelogram. Line MN meets the circumcircle of ABC at R and S. Prove that the circumcircle of triangle RPS is tangent to BC.
4	Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $$f(n+1) > \frac{f(n) + f(f(n))}{2}$$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of strictly positive integers.
–	Day II
1	Let Γ be a circle with center O and AE be a diameter. Point D lies on segment OE and point B is the midpoint of one of the arcs $\widehat{AE}$ of Γ. Construct point C such that $ABCD$ is a parallelogram. Lines EB and CD meet at F. Line OF meets the minor arc $\widehat{EB}$ at I. Prove that EI bisects $\angle BEC$.
2	Let S be a set of positive real numbers with five elements such that for any distinct $a,\ b,\ c$ in S, the number $ab + bc + ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.
3	Show that it is possible to write a $n \times n$ array of non-negative numbers (not necessarily distinct) such that the sums of entries on each row and each column are pairwise distinct perfect squares.

4 Aws plays a solitaire game on a fifty-two card deck: whenever two cards of the same color are adjacent, he can remove them. Aws wins the game if he removes all the cards. If Aws starts with the cards in a random order, what is the probability for him to win?

– Day III

1 A *perfect number* is an integer that equals half the sum of its positive divisors. For example, because $2 \cdot 28 = 1 + 2 + 4 + 7 + 14 + 28$, 28 is a perfect number.

- **(a)** A *square-free* integer is an integer not divisible by a square of any prime number. Find all square-free integers that are perfect numbers.

- **(b)** Prove that no perfect square is a perfect number.

2 Determine all functions $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ and

$$f(x) = 1 + 5f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) - 6f\left(\left\lfloor \frac{x}{4} \right\rfloor\right)$$

for all $x > 0$.

3 There are 2015 coins on a table. For $i = 1, 2, \dots, 2015$ in succession, one must turn over exactly i coins. Prove that it is always possible either to make all of the coins face up or to make all of the coins face down, but not both.

4 Let ω_1 and ω_2 with center O_1 and O_2 respectively, meet at points A and B. Let X and Y be points on ω_1. Lines XA and YA meet ω_2 at Z and W, respectively, such that A lies between X and Z and between Y and W. Let M be the midpoint of O_1O_2, S be the midpoint of XA and T be the midpoint of WA. Prove that $MS = MT$ if and only if $X,\ Y,\ Z$ and W are concyclic.

– Day IV

1 Let $a_1, \dots, a_n$ be a non increasing sequence of positive real numbers. Prove that

$$\sqrt{a_1^2 + a_2^2 + \dots + a_n^2} \le a_1 + \frac{a_2}{\sqrt{2} + 1} + \dots + \frac{a_n}{\sqrt{n} + \sqrt{n-1}}.$$

When does equality hold?

2 In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of his points against the other nine of the ten). What was the total number of players in the tournament?

3 We are given a lattice and two pebbles A and B that are placed at two lattice points. At each step we are allowed to relocate one of the pebbles to another lattice point with the condition that the distance between pebbles is preserved. Is it possible after finite number of steps to switch positions of the pebbles?

4 Points $A_1,\ B_1,\ C_1$ lie on the sides $BC,\ AC$ and AB of a triangle ABC, respectively, such that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let $I_A,\ I_B,\ I_C$ be the incenters of triangles $AB_1C_1,\ A_1BC_1$ and A_1B_1C respectively. Prove that the circumcenter of triangle $I_AI_BI_C$, is the incenter of triangle ABC.