Saudi Arabia Team Selection Test for Balkan Math Olympiad 2014

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– Day I

1 A positive proper divisor is a positive divisor of a number, excluding itself. For positive integers $n \ge 2$, let $f(n)$ denote the number that is one more than the largest proper divisor of n. Determine all positive integers n such that $f(f(n)) = 2$.

2 Prove that among any 16 perfect cubes we can always find two cubes whose difference is divisible by 91.

3 Let $n \ge 2$ be a positive integer, and write in a digit form

$$\frac{1}{n} = 0.a_1a_2\dots.$$

Suppose that $n = a_1 + a_2 + \cdots$. Determine all possible values of n.

4 Let ABC be a triangle with $\angle B \le \angle C$, I its incenter and D the intersection point of line AI with side BC. Let M and N be points on sides BA and CA, respectively, such that $BM = BD$ and $CN = CD$. The circumcircle of triangle CMN intersects again line BC at P. Prove that quadrilateral $DIMP$ is cyclic.

5 Find all positive integers n such that

$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$

– Day II

1 Find the minimum of $\sum\limits_{k=0}^{40} \left(x+\frac{k}{2}\right)^2$ where x is a real numbers

2 Let $\mathbb{N}$ denote the set of positive integers, and let S be a set. There exists a function $f : \mathbb{N} \to S$ such that if x and y are a pair of positive integers with their difference being a prime number, then $f(x) \ne f(y)$. Determine the minimum number of elements in S.

3 Let $ABCD$ be a parallelogram. A line ℓ intersects lines $AB,\ BC,\ CD,\ DA$ at four different points $E,\ F,\ G,\ H,$ respectively. The circumcircles of triangles AEF and AGH intersect again at P. The circumcircles of triangles CEF and CGH intersect again at Q. Prove that the line PQ bisects the diagonal BD.

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4 Let n be an integer greater than 2. Consider a set of n different points, with no three collinear, in the plane. Prove that we can label the points $P_1,\ P_2, \dots, P_n$ such that $P_1P_2 \dots P_n$ is not a self-intersecting polygon. (*A polygon is self-intersecting if one of its side intersects the interior of another side. The polygon is not necessarily convex*)

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5 Let $n > 3$ be an odd positive integer not divisible by 3. Determine if it is possible to form an $n \times n$ array of numbers such that

- **(a)** the set of the numbers in each row is a permutation of $0, 1, \dots, n-1$;
the set of the numbers in each column is a permutation of $0, 1, \dots, n-1$;

- **(b)** the board is *totally non-symmetric*: for $1 \le i < j \le n$ and $1 \le i' < j' \le n$, if $(i, j) \ne (i', j')$ then $(a_{i,j}, a_{j,i}) \ne (a_{i',j'}, a_{j',i'})$ where $a_{i,j}$ denotes the entry in the i^{th} row and j^{th} column.

– Day III

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1 Find all functions $f : \mathbb{N} \to (0, \infty)$ such that $f(4) = 4$ and

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)},\ \forall n \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, \dots\}$ is the set of positive integers.

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2 Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points $Q,\ X,\ Y,\ Z$ are collinear.

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3 Let a, b be two nonnegative real numbers and n a positive integer. Prove that

$$\left(1 - 2^{-n}\right) \left|a^{2^n} - b^{2^n}\right| \ge \sqrt{ab} \left|a^{2^n - 1} - b^{2^n - 1}\right|.$$

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4 Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function such that $f(1) = 2,\ f(2) = 4$ and

$$f(f(m) + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}$. Prove that $f(n) = n + 2$ for all $n \ge 2$.

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5 Let ABC be a triangle. Circle Ω passes through points B and C. Circle ω is tangent internally to Ω and also to sides AB and AC at $T,\ P,$ and Q, respectively. Let M be midpoint of arc $\widehat{BC}$ (containing T) of Ω. Prove that lines $PQ,\ BC,$ and MT are concurrent.
