Dutch BxMO Team Selection Test 2017

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by Medjl

1 Let n be an even positive integer. A sequence of n real numbers is called complete if for every integer m with $1 \le m \le n$ either the sum of the first m terms of the sum or the sum of the last m terms is integral. Determine
the minimum number of integers in a complete sequence of n numbers.

2 Let define a function $f : \mathbb{N} \to \mathbb{Z}$ such that :
$i) f(p) = 1$ for all prime numbers p.
$ii) f(xy) = xf(y) + yf(x)$ for all positive integers x, y
find the smallest $n \ge 2016$ such that $f(n) = n$

3 Let ABC be a triangle with $\angle A = 90$ and let D be the orthogonal projection of A onto BC. The midpoints of AD and AC are called E and F, respectively. Let M be the circumcentre of BEF. Prove that AC and BM are parallel.

4 A quadruple $(a; b; c; d)$ of positive integers with $a \le b \le c \le d$ is called good if we can colour each integer red, blue, green or purple, in such a way that i of each a consecutive integers at least one is coloured red; ii of each b consecutive integers at least one is coloured blue; iii of each c consecutive integers at least one is coloured green; $iiii$ of each d consecutive integers at least one is coloured purple.
Determine all good quadruples with $a = 2$.

5 Determine all pairs of prime numbers $(p; q)$ such that $p^2 + 5pq + 4q^2$ is the square of an integer.