Dutch IMO Team Selection Test 2017

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– Day I

1 Let n be a positive integer. Suppose that we have disks of radii $1, 2, ..., n$. Of each size there are two disks: a transparent one and an opaque one. In every disk there is a small hole in the centre, with which we can stack the
disks using a vertical stick. We want to make stacks of disks that satisfy the following conditions:
$i)$ Of each size exactly one disk lies in the stack.
$ii)$ If we look at the stack from directly above, we can see the edges of all of the n disks in the stack. (So if there is an opaque disk in the stack,no smaller disks may lie beneath it.)
Determine the number of distinct stacks of disks satisfying these conditions.
(Two stacks are distinct if they do not use the same set of disks, or, if they do use the same set of disks and the orders in which the disks occur are different.)

2 Let $n \geq 4$ be an integer. Consider a regular $2n-$gon for which to every vertex, an integer is assigned, which we call the value of said vertex. If four distinct vertices of this $2n-$gon form a rectangle, we say that the sum of the values of these vertices is a rectangular sum.
Determine for which (not necessarily positive) integers m the integers $m+1, m+2, ..., m+2n$ can be assigned to the vertices (in some order) in such a way that every rectangular sum is a prime number. (Prime numbers are positive by definition.)

3 let x, y be non-zero reals such that : $x^3 + y^3 + 3x^2y^2 = x^3y^3$
find all values of $\frac{1}{x} + \frac{1}{y}$

4 Let ABC be a triangle, let M be the midpoint of AB, and let N be the midpoint of CM. Let X be a point satisfying both $\angle XMC = \angle MBC$ and $\angle XCM = \angle MCB$ such that X and B lie on opposite sides of CM. Let ω be the circumcircle of triangle AMX. (a) Show that CM is tangent to ω. (b) Show that the lines NX and AC intersect on ω

– Day II

1 Let a, b, c be distinct positive integers, and suppose that $p = ab + bc + ca$ is a prime number.
(a) Show that $a^2, b^{\cdot} c^2$ give distinct remainders after division by p.
(b) Show that a^3, b^3, c^3 give distinct remainders after division by p.

2 The incircle of a non-isosceles triangle ABC has centre I and is tangent to BC and CA in D and E, respectively. Let H be the orthocentre of ABI, let K be the intersection of AI and BH

and let L be the intersection of BI and AH. Show that the circumcircles of DKH and ELH intersect on the incircle of ABC.

3 Let $k > 2$ be an integer. A positive integer l is said to be $k-pable$ if the numbers $1, 3, 5, ..., 2k-1$ can be partitioned into two subsets A and B in such a way that the sum of the elements of A is exactly l times as large as the sum of the elements of B.
Show that the smallest $k-pable$ integer is coprime to k.

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(y + 1)f(x) + f(xf(y) + f(x + y)) = y$$

for all $x, y \in \mathbb{R}$.

– Day III

1 A circle ω with diameter AK is given. The point M lies in the interior of the circle, but not on AK. The line AM intersects ω in A and Q. The tangent to ω at Q intersects the line through M perpendicular to AK, at P. The point L lies on ω, and is such that PL is tangent to ω and $L \neq Q$.
Show that K, L, and M are collinear.

2 let $a_1, a_2, ...a_n$ a sequence of real numbers such that $a_1 + ... + a_n = 0$.
define $b_i = a_1 + a_2 + ...a_i$ for all $1 \leq i \leq n$ .suppose $b_i(a_{j+1} - a_{i+1}) \geq 0$ for all $1 \leq i \leq j \leq n-1$.
Show that

$$\max_{1 \leq l \leq n} |a_l| \geq \max_{1 \leq m \leq n} |b_m|$$

3 Compute the product of all positive integers n for which $3(n! + 1)$ is divisible by $2n - 5$.

4 Let $n \geq 2$ be an integer. Find the smallest positive integer m for which the following holds: given n points in the plane, no three on a line, there are m lines such that no line passes through any of the given points, and
for all points $X \neq Y$ there is a line with respect to which X and Y lie on opposite sides