Dutch IMO TST Team Selection Test 2013

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– Day 1

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1 Determine all 4-tuples (a, b, c, d) of real numbers satisfying the following four equations: $\begin{cases} ab + c + d = 3 \\ bc + d + a = 5 \\ cd + a + b = 2 \\ da + b + c = 6 \end{cases}$

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2 Determine all integers n for which $\frac{4n-2}{n+5}$ is the square of a rational number.

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3 Fix a triangle ABC. Let Γ_1 the circle through B, tangent to edge in A. Let Γ_2 the circle through C tangent to edge AB in A. The second intersection of Γ_1 and Γ_2 is denoted by D. The line AD has second intersection E with the circumcircle of $\triangle\ ABC$. Show that D is the midpoint of the segment AE.

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4 Let $n \geq 3$ be an integer, and consider a $n \times n$-board, divided into n^2 unit squares. For all $m \geq 1$, arbitrarily many $1 \times m$-rectangles (type I) and arbitrarily many $m \times 1$-rectangles (type II) are available. We cover the board with N such rectangles, without overlaps, and such that every rectangle lies entirely inside the board. We require that the number of type I rectangles used is equal to the number of type II rectangles used.(Note that a 1×1-rectangle has both types.) What is the minimal value of N for which this is possible?

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5 Let a, b, and c be positive real numbers satisfying $abc = 1$.
Show that $a + b + c \geq \sqrt{\frac{1}{3}(a+2)(b+2)(c+2)}$

– Day 2

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1 Show that $\sum_{n=0}^{2013} \frac{4026!}{(n!(2013-n)!)^2}$ is a perfect square.

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2 Let P be the point of intersection of the diagonals of a convex quadrilateral $ABCD$.Let X, Y, Z be points on the interior of AB, BC, CD respectively such that $\frac{AX}{XB} = \frac{BY}{YC} = \frac{CZ}{ZD} = 2$. Suppose that XY is tangent to the circumcircle of $\triangle CYZ$ and that YZ is tangent to the circumcircle of $\triangle BXY$.Show that $\angle APD = \angle XYZ$.

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3 Fix a sequence $a_1, a_2, a_3 \ldots$ of integers satisfying the following condition:for all prime numbers p and all positive integers k,we have $a_{pk+1} = pa_k - 3a_p + 13$.Determine all possible values of

a_{2013}.

4 Determine all positive integers $n \geq 2$ satisfying $i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod{2}$ for all i and j with $0 \leq i \leq j \leq n$.

5 Let $ABCDEF$ be a cyclic hexagon satisfying $AB \perp BD$ and $BC = EF$.Let P be the intersection of lines BC and AD and let Q be the intersection of lines EF and AD.Assume that P and Q are on the same side of D and A is on the opposite side.Let S be the midpoint of AD.Let K and L be the incentres of $\triangle BPS$ and $\triangle EQS$ respectively.Prove that $\angle KDL = 90^0$.