Gulf Mathematical Olympiad 2015

www.artofproblemsolving.com/community/c930578

by parmenides51

1 a) Suppose that n is an odd integer. Prove that $k(n-k)$ is divisible by 2 for all positive integers k.

b) Find an integer k such that $k(100-k)$ is not divisible by 11.

c) Suppose that p is an odd prime, and n is an integer.
Prove that there is an integer k such that $k(n-k)$ is not divisible by p.

d) Suppose that p,q are two different odd primes, and n is an integer.
Prove that there is an integer k such that $k(n-k)$ is not divisible by any of p,q.

---

2 a) Let UVW , $U'V'W'$ be two triangles such that $VW = V'W', UV = U'V', \angle WUV = \angle W'U'V'$. Prove that the angles $\angle VWU, \angle V'W'U'$ are equal or supplementary.

b) ABC is a triangle where $\angle A$ is **obtuse**. take a point P inside the triangle , and extend AP, BP, CP to meet the sides BC, CA, AB in K, L, M respectively. Suppose that $PL = PM$.
1) If AP bisects $\angle A$, then prove that $AB = AC$.
2) Find the angles of the triangle ABC if you know that AK, BL, CM are angle bisectors of the triangle ABC and that $2AK = BL$.

---

3 We have a large supply of black, white, red and green hats.
And we want to give 8 of these hats to 8 students that are sitting around a round table.
Find the number of ways of doing that in each of these cases (assuming for the purposes of this problem that students will notchange their places, and that hats of the same color are identical)
a) Each hat to be used must be either red or green.
b) Exactly two hats of each color are to be used
c) Exactly two hats of each color are to be used, and every two hats of the same color are to be given to two adjacent students.
d) Exactly two hats of each color are to be used, and no two hats of the same color are to be given to two adjacent students.
e) There are no restrictions on the number of hats of each color that are to be used, but no two hats of the same color are to be given to two adjacent students.

---

4 a) We have a geometric sequence of 3 terms. If the sum of these terms is 26 , and their sum of squares is 364 , find the terms of the sequence.

b) Suppose that a, b, c, u, v, w are positive real numbers , and each of a, b, c and u, v, w are geometric sequences. Suppose also that $a+u, b+v, c+w$ are an arithmetic sequence. Prove that

$a = b = c$ and $u = v = w$

c) Let a, b, c, d be real numbers (not all zero), and let $f(x, y, z)$ be the polynomial in three variables defined by

$$f(x, y, z) = axyz + b(xy + yz + zx) + c(x + y + z) + d$$

.Prove that $f(x, y, z)$ is reducible if and only if a, b, c, d is a geometric sequence.