Gulf Mathematical Olympiad 2014

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by m2121, parmenides51

1 A sequence $a_0, a_1, a_2, \cdots$ satisfies the conditions $a_0 = 0$, $a_{n-1}^2 - a_{n-1} = a_n^2 + a_n$
1) determine the two possible values of a_1 . then determine all possible values of a_2 .
2)for each n, prove that $a_{n+1} = a_n + 1$ or $a_{n+1} = -a_n$
3)Describe the possible values of a_{1435}
4)Prove that the values that you got in (3) are correct

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2 Ahmad and Salem play the following game. Ahmad writes two integers (not necessarily different) on a board. Salem writes their sum and product. Ahmad does the same thing: he writes the sum and product of the two numbers which Salem has just written.
They continue in this manner, not stopping unless the two players write the same two numbers one after the other (for then they are stuck!). The order of the two numbers which each player writes is not important.
Thus if Ahmad starts by writing 3 and -2, the first five moves (or steps) are as shown:
(a) Step 1 (Ahmad) 3 and -2
(b) Step 2 (Salem) 1 and -6
(c) Step 3 (Ahmad) -5 and -6
(d) Step 4 (Salem) -11 and 30
(e) Step 5 (Ahmad) 19 and -330

(i) Describe all pairs of numbers that Ahmad could write, and ensure that Salem must write the same numbers, and so the game stops at step 2.
(ii) What pair of integers should Ahmad write so that the game finishes at step 4?
(iii) Describe all pairs of integers which Ahmad could write at step 1, so that the game will finish after finitely many steps.
(iv) Ahmad and Salem decide to change the game. The first player writes three numbers on the board, u, v and w. The second player then writes the three numbers $u+v+w, uv+vw+wu$ and uvw, and they proceed as before, taking turns, and using this new rule describing how to work out the next three numbers. If Ahmad goes first, determine all collections of three numbers which he can write down, ensuring that Salem has to write the same three numbers at the next step.

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3 (i) ABC is a triangle with a right angle at A, and P is a point on the hypotenuse BC.
The line AP produced beyond P meets the line through B which is perpendicular to BC at U.
Prove that $BU = BA$ if, and only if, $CP = CA$.
(ii) A is a point on the semicircle CB, and points X and Y are on the line segment BC.
The line AX, produced beyond X, meets the line through B which is perpendicular to BC at U.

Also the line AY, produced beyond Y, meets the line through C which is perpendicular to BC at V.
Given that $BY = BA$ and $CX = CA$, determine the angle $\angle VAU$.

4 The numbers from 1 to 64 must be written on the small squares of a chessboard, with a different number in each small square. Consider the 112 numbers you can make by adding the numbers in two small squares which have a common edge. Is it possible to write the numbers in the squares so that these 112 sums are all different?