Romania National Olympiad 2015

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–	Grade 9
1	Show that among the square roots of the first 2015 natural numbers, we cannot choose an arithmetic sequence composed of 45 elements.
2	A quadratic function has the property that for any interval of length $1,$ the length of its image is at least $1.$ Show that for any interval of length $2,$ the length of its image is at least $4.$
3	Let be a point P in the interior of a triangle $ABC.$ The lines AP, BP, CP meet $BC, AC,$ respectively, AB at $A_1, B_1,$ respectively, $C_1.$ If $$\mathcal{A}_{PBA_1}+\mathcal{A}_{PCB_1}+\mathcal{A}_{PAC_1}=\frac{1}{2}\mathcal{A}_{ABC},$$ show that P lies on a median of $ABC.$ *$\mathcal{A}$ denotes area.*
4	Let $a,b,c,d\geq 0$ real numbers so that $a+b+c+d=1$.Prove that $\sqrt{a+\frac{(b-c)^2}{6}+\frac{(c-d)^2}{6}+\frac{(d-b)^2}{6}}+\sqrt{b}+\sqrt{c}+\sqrt{d}\leq 2.$
–	Grade 10
1	Find all triplets (a,b,c) of nonzero complex numbers having the same absolute value and which verify the equality: $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=-1$$
2	Consider a natural number n for which it exist a natural number k and k distinct primes so that $n=p_1\cdot p_2\cdots p_k.$ **a)** Find the number of functions $f:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ that have the property that $f(1)\cdot f(2)\cdots f(n)$ divides $n.$ **b)** If $n=6,$ find the number of functions $f:\{1,2,3,4,5,6\}\longrightarrow\{1,2,3,4,5,6\}$ that have the property that $f(1)\cdot f(2)\cdot f(3)\cdot f(4)\cdot f(5)\cdot f(6)$ divides $36.$

3 Find all functions $f, g : \mathbb{Q} \longrightarrow \mathbb{Q}$ that verify the relations

$$\begin{cases} f(g(x)+g(y)) = f(g(x)) + y \\ g(f(x)+f(y)) = g(f(x)) + y \end{cases},$$

for all $x, y \in \mathbb{Q}$.

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4 Let be a finite set A of real numbers, and define the sets $S_{\pm} = \{x \pm y | x, y \in A\}$.
Show that $|A| \cdot |S_-| \le |S_+|^2$.

– Grade 11

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1 Find all differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that verify the conditions: (i) $\forall x \in \mathbb{Z} \quad f'(x) = 0$

(ii) $\forall x \in \mathbb{R} \quad f'(x) = 0 \implies f(x) = 0$

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2 Let be a 5×5 complex matrix A whose trace is $0,$ and such that $I_5 - A$ is invertible.
Prove that $A^5 \neq I_5.$

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3 Let be two nonnegative real numbers a, b with $b > a,$ and a sequence $(x_n)_{n\ge 1}$ of real numbers such that the sequence $\left(\frac{x_1+x_2+\cdots+x_n}{n^a}\right)_{n\ge 1}$ is bounded.

Show that the sequence $\left(x_1 + \frac{x_2}{2^b} + \frac{x_3}{3^b} + \cdots + \frac{x_n}{n^b}\right)_{n\ge 1}$ is convergent.

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4 Let be three natural numbers k, m, n an $m \times n$ matrix $A,$ an $n \times m$ matrix $B,$ and k complex numbers $a_0, a_1, \ldots, a_k$ such that the following conditions hold.

(i) $m \ge n \ge 2$ (ii) $a_0 I_m + a_1 AB + a_2 (AB)^2 + \cdots + a_k (AB)^k = O_m$ (iii) $a_0 I_m + a_1 BA + a_2 (BA)^2 + \cdots + a_k (BA)^k \neq O_n$

Prove that $a_0 = 0.$

– Grade 12

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1 Let be a ring that has the property that all its elements are the product of two idempotent elements of it. Show that:
a) 1 is the only unit of this ring.
b) this ring is Boolean.

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2 Show that the set of all elements minus 0 of a finite division ring that has at least 4 elements can be partitioned into two nonempty sets A, B having the property that

$$\sum_{x\in A} x = \prod_{y \in B} y.$$

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3 Let be the set $\mathcal{C} = \left\{ f : [0,1] \longrightarrow \mathbb{R} \middle| \exists f'' \middle|_{[0,1]} \quad \exists x_1, x_2 \in [0,1] \quad x_1 \neq x_2 \wedge (f(x_1) = f(x_2) = 0 \vee f(x_1) = f'\right.$

and $f^* \in \mathcal{C}$ such that $\int_0^1 |f^*(x)|\, dx = \sup_{f\in \mathcal{C}} \int_0^1 |f(x)|\, dx.$

Find $\int_0^1 |f^*(x)|\, dx$ and describe f^*.

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4 Find all non-constant polynoms $f \in \mathbb{Q}[X]$ that don't have any real roots in the interval $[0,1]$ and for which there exists a function $\xi : [0,1] \longrightarrow \mathbb{Q}[X] \times \mathbb{Q}[X], \xi(x) := (g_x, h_x)$ such that $h_x(x) \neq 0$ and $\int_0^x \frac{dt}{f(t)} = \frac{g_x(x)}{h_x(x)}$, for all $x \in [0,1]$.
