Romania National Olympiad 2015

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–	Grade 9
1	The orthocenter H of a triangle ABC is distinct from its vertices and its circumcenter O. M, N, P are the circumcenters of the triangles HBC, HCA, respectively, HAB. Prove that AM, BN, CP and OH are concurrent.
2	Let be a natural number $n \geq 2$ and n positive real numbers $a_1, a_n, \ldots, a_n$ that satisfy the inequalities $$\sum_{j=1}^{i} a_j \leq a_{i+1}, \quad \forall i \in \{1, 2, \ldots, n-1\}.$$ Prove that $$\sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}} \leq n/2.$$
3	We say that a rational number is *spheric* if it is the sum of three squares of rational numbers (not necessarily distinct). Prove that: **a)** 7 is not spheric. **b)** a rational spheric number raised to the power of any natural number greater than 1 is spheric.
4	Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the inequality $$f(a^2) - f(b^2) \leq (f(a) + b)(a - f(b)),$$ for all $a, b \in \mathbb{R}$.
–	Grade 10
1	Let be a natural number $n \geq 2$ and n positive real numbers $a_1, a_2, \ldots, a_n$ whose product is 1. Prove that the function $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}, \quad f(x) = \prod_{i=1}^{n} \left(1 + a_i^x\right),$ is nondecreasing.
2	Let be a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the conditions: $$\begin{cases} f(x+y) & \leq & f(x) + f(y) \\ f(tx + (1-t)y) & \leq & t(f(x)) + (1-t)f(y) \end{cases},$$ for all real numbers x, y, t with $t \in [0, 1]$.

Prove that:
a) $f(b)+f(c) \leq f(a)+f(d)$, for any real numbers a, b, c, d such that $a \leq b \leq c \leq d$ and $d-c=b-a$.
b) for any natural number $n \geq 3$ and any n real numbers $x_1, x_2, \ldots, x_n$, the following inequality holds.

$$f\left(\sum_{1\le i\le n} x_i\right)+(n-2)\sum_{1\le i\le n} f\left(x_i\right) \geq \sum_{1\le i<j\le n} f\left(x_i+x_j\right)$$

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3 **a)** Let be two nonzero complex numbers a, b. Show that the area of the triangle formed by the representations of the affixes $0, a, b$ in the complex plane is $\frac{1}{4}\left|\overline{a}b-a\overline{b}\right|$.

b) Let be an equilateral triangle ABC, its circumcircle $\mathcal{C}$, its circumcenter O, and two distinct points P_1, P_2 in the interior of $\mathcal{C}$. Prove that we can form two triangles with sides P_1A, P_1B, P_1C, respectively, P_2A, P_2B, P_2C, whose areas are equal if and only if $OP_1=OP_2$.

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4 In order to study a certain ancient language, some researchers formatted its discovered words into expressions formed by concatenating letters from an alphabet containing only two letters. Along the study, they noticed that any two distinct words whose formatted expressions have an equal number of letters, greater than 2, differ by at least three letters.
Prove that if their observation holds indeed, then the number of formatted expressions that have $n\geq 3$ letters is at most $\left[\frac{2^n}{n+1}\right]$.

– Grade 11

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1 Let be a 2×2 real matrix A that has the property that $\left|A^d-I_2\right|=\left|A^d+I_2\right|$, for all $d\in\{2014, 2016\}$.
Prove that $\left|A^n-I_2\right|=\left|A^n+I_2\right|$, for any natural number n.

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2 Consider a natural number, $n\geq 2$, and three $n\times n$ complex matrices A, B, C such that A is invertible, B is formed by replacing the first line of A with zeroes, and C is formed by putting the last $n-1$ lines of A above a line of zeroes. Prove that:

a) rank $\left(A^{-1}B\right)=$ rank $\left(\left(A^{-1}B\right)^2\right)=\cdots=$ rank $\left(\left(A^{-1}B\right)^n\right)$

b) rank $\left(A^{-1}C\right)>$ rank $\left(\left(A^{-1}C\right)^2\right)>\cdots>$ rank $\left(\left(A^{-1}C\right)^n\right)$

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3 Let be a real number a, and a function $f:\mathbb{R}_{>0}\longrightarrow\mathbb{R}_{>0}$. Show that the following relations are equivalent.

(i) $\varepsilon\in\mathbb{R}_{>0}\implies\left(\lim_{x\to\infty}\frac{f(x)}{x^{a+\varepsilon}}=0\wedge\lim_{x\to\infty}\frac{f(x)}{x^{a-\varepsilon}}=\infty\right)$ (ii) $\lim_{x\to\infty}\frac{\ln f(x)}{\ln x}=a$

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4 Find all functions, $f:\mathbb{R}\longrightarrow\mathbb{R}$, that have the properties that f^2 is differentiable and $f=\left(f^2\right)'$.

– Grade 12

1 Prove that there exists an unique sequence $(c_n)_{n\geq 1}$ of real numbers from the interval $(0,1)$ such that
$$\int_0^1 \frac{dx}{1+x^m} = \frac{1}{1+c_m^m},$$
for all natural numbers m, and calculate $\lim_{k\to\infty} kc_k^k$.

Radu Pop

2 Let A be a ring and let D be the set of its non-invertible elements. If $a^2=0$ for any $a\in D$, prove that:
a) $axa=0$ for all $a\in D$ and $x\in A$;
b) if D is a finite set with at least two elements, then there is $a\in D$, $a\neq 0$, such that $ab=ba=0$, for every $b\in D$.

Ioan Băetu

3 Let be a real number a, and a nondecreasing function $f:\mathbb{R}\longrightarrow\mathbb{R}$. Prove that f is continuous in a if and only if there exists a sequence $(a_n)_{n\geq 1}$ of real positive numbers such that
$$\int_a^{a+a_n} f(x)dx + \int_a^{a-a_n} f(x)dx \leq \frac{a_n}{n},$$
for all natural numbers n.

Dan Marinescu

4 Let K be a finite field with q elements, $q\geq 3$. We denote by M the set of polynomials in $K[X]$ of degree $q-2$ whose coefficients are nonzero and pairwise distinct. Find the number of polynomials in M that have $q-2$ distinct roots in K.

Marian Andronache