Dutch BxMO/EGMO Team Selection Test 2012

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by parmenides51

1 Do there exist quadratic polynomials $P(x)$ and $Q(x)$ with real coeffcients such that the polynomial $P(Q(x))$ has precisely the zeros $x = 2, x = 3, x = 5$ and $x = 7$?

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2 Let $\triangle ABC$ be a triangle and let X be a point in the interior of the triangle. The second intersection points of the lines XA, XB and XC with the circumcircle of $\triangle ABC$ are P, Q and R. Let U be a point on the ray XP (these are the points on the line XP such that P and U lie on the same side of X). The line through U parallel to AB intersects BQ in V . The line through U parallel to AC intersects CR in W. Prove that Q, R, V , and W lie on a circle.

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3 Find all pairs of positive integers (x, y) for which $x^3 + y^3 = 4(x^2y + xy^2 - 5)$.

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4 Let $ABCD$ a convex quadrilateral (this means that all interior angles are smaller than 180^o), such that there exist a point M on line segment AB and a point N on line segment BC having the property that AN cuts the quadrilateral in two parts of equal area, and such that the same property holds for CM.
Prove that MN cuts the diagonal BD in two segments of equal length.

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5 Let A be a set of positive integers having the following property:
for each positive integer n exactly one of the three numbers $n, 2n$ and $3n$ is an element of A.
Furthermore, it is given that $2 \in A$. Prove that $13824 \notin A$.
