Dutch BxMO Team Selection Test 2010

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1 Let $ABCD$ be a trapezoid with $AB//CD$, $2|AB| = |CD|$ and $BD \perp BC$. Let M be the midpoint of CD and let E be the intersection BC and AD. Let O be the intersection of AM and BD. Let N be the intersection of OE and AB.
(a) Prove that $ABMD$ is a rhombus.
(b) Prove that the line DN passes through the midpoint of the line segment BE.

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2 Find all functions $f : R \to R$ satisfying $f(x)f(y) = f(x + y) + xy$ for all $x, y \in R$.

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3 Let N be the number of ordered 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ of positive integers satisfying

$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} = 1$

Is N even or odd?

Oh and **HINTS ONLY**, please do not give full solutions. Thanks.

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4 The two circles Γ_1 and Γ_2 intersect at P and Q. The common tangent that's on the same side as P, intersects the circles at A and B,respectively. Let C be the second intersection with Γ_2 of the tangent to Γ_1 at P, and let D be the second intersection with Γ_1 of the tangent to Γ_2 at Q. Let E be the intersection of AP and BC, and let F be the intersection of BP and AD. Let M be the image of P under point reflection with respect to the midpoint of AB. Prove that $AMBEQF$ is a cyclic hexagon.

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5 For any non-negative integer n, we say that a permutation $(a_0, a_1, ..., a_n)$ of $\{0, 1, ..., n\}$ is quadratic if $k + a_k$ is a square for $k = 0, 1, ..., n$. Show that for any non-negative integer n, there exists a quadratic permutation of $\{0, 1, ..., n\}$.
