NIMO Summer Contest 2017

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by parmenides51, djmathman

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1 Let x be the answer to this question. Find the value of $2017 - 2016x$.

Proposed by Michael Tang

2 Joy has 33 thin rods, one each of every integer length from 1 cm through 30 cm, and also three more rods with lengths 3 cm, 7 cm, and 15 cm. She places those three rods on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

Proposed by Michael Tang

3 If p, q, and r are nonzero integers satisfying

$$p^2 + q^2 = r^2,$$

compute the smallest possible value of $(p+q+r)^2$.

Proposed by David Altizio

4 The square $BCDE$ is inscribed in circle ω with center O. Point A is the reflection of O over B. A "hook" is drawn consisting of segment AB and the major arc $\widehat{BE}$ of ω (passing through C and D). Assume $BCDE$ has area 200. To the nearest integer, what is the length of the hook?

Proposed by Evan Chen

5 Find the smallest positive integer n for which the number

$$A_n = \prod_{k=1}^{n} \binom{k^2}{k} = \binom{1}{1}\binom{4}{2}\cdots\binom{n^2}{n}$$

ends in the digit 0 when written in base ten.

Proposed by Evan Chen

6 Let $P = (-2, 0)$. Points P, Q, R lie on the graph of the function $y = x^3 - 3x + 2$ such that Q is the midpoint of segment PR. Compute PR^2.

Proposed by David Altizio

7 Let S be the maximum possible value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$. Given that S can be written in the form m/n where m, n are coprime positive integers, find $100m+n$.

Proposed by Kaan Dokmeci

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8 Konsistent Karl is taking this contest. He can solve the first five problems in one minute each, the next five in two minutes each, and the last five in three minutes each. What is the maximum possible score Karl can earn? (Recall that this contest is 15 minutes long, there are 15 problems, and the nth problem is worth n points. Assume that entering answers and moving between or skipping problems takes no time.)

Proposed by Michael Tang

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9 Let P be a cubic monic polynomial with roots a, b, and c. If $P(1)=91$ and $P(-1)=-121$, compute the maximum possible value of

$$\frac{ab+bc+ca}{abc+a+b+c}.$$

Proposed by David Altizio

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10 In triangle ABC we have $AB=36$, $BC=48$, $CA=60$. The incircle of ABC is centered at I and touches AB, AC, BC at M, N, D, respectively. Ray AI meets BC at K. The radical axis of the circumcircles of triangles MAN and KID intersects lines AB and AC at L_1 and L_2, respectively. If $L_1L_2=x$, compute x^2.

Proposed by Evan Chen

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11 Let $a, b, c, p, q, r > 0$ such that (a, b, c) is a geometric progression and (p, q, r) is an arithmetic progression. If

$$a^pb^qc^r=6 \quad \text{and} \quad a^qb^rc^p=29$$

then compute $\lfloor a^rb^pc^q \rfloor$.

Proposed by Michael Tang

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12 Triangle ABC has $AB=2$, $BC=3$, $CA=4$, and circumcenter O. If the sum of the areas of triangles AOB, BOC, and COA is $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c, where $\gcd(a,c)=1$ and b is not divisible by the square of any prime, find $a+b+c$.

Proposed by Michael Tang

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13 We say that $1 \leq a \leq 101$ is a quadratic polynomial residue modulo 101 with respect to a quadratic polynomial $f(x)$ with integer coefficients if there exists an integer b such that $101 \mid a - f(b)$. For a quadratic polynomial f, we define its quadratic residue set as the set of quadratic residues modulo 101 with respect to $f(x)$. Compute the number of quadratic residue sets.

Proposed by Michael Ren

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14 Let x, y, z be real numbers such that $x + y + z = -2$ and

$$\begin{aligned}&(x^2 + xy + y^2)(y^2 + yz + z^2)\\&+ (y^2 + yz + z^2)(z^2 + zx + x^2)\\&+ (z^2 + zx + x^2)(x^2 + xy + y^2)\\&= 625 + \tfrac{3}{4}(xy + yz + zx)^2.\end{aligned}$$

Compute $|xy + yz + zx|$.

Proposed by Michael Tang

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15 For all positive integers n, denote by $\sigma(n)$ the sum of the positive divisors of n and $\nu_p(n)$ the largest power of p which divides n. Compute the largest positive integer k such that 5^k divides

$$\sum_{d \mid N} \nu_3(d!)(-1)^{\sigma(d)},$$

where $N = 6^{1999}$.

Proposed by David Altizio
