Hong Kong Team Selection Test 2020

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Test 1 August 24, 2019

1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n the following is valid: If $d_1, d_2, \ldots, d_s$ are all the positive divisors of n, then

$$f(d_1)f(d_2)\ldots f(d_s) = n.$$

2 Let D be an arbitrary point inside ΔABC. Let Γ be the circumcircle of ΔBCD. The external angle bisector of $\angle ABC$ meets Γ again at E. The external angle bisector of $\angle ACB$ meets Γ again at F. The line EF meets the extension of AB and AC at P and Q respectively. Prove that the circumcircles of ΔBFP and ΔCEQ always pass through the same fixed point regardless of the position of D. (Assume all the labelled points are distinct.)

3 Given a list of integers $2^1+1, 2^2+1, \ldots, 2^{2019}+1$, Adam chooses two different integers from the list and computes their greatest common divisor. Find the sum of all possible values of this greatest common divisor.

4 Find the total number of primes $p < 100$ such that $\lfloor (2+\sqrt{5})^p \rfloor - 2^{p+1}$ is divisible by p. Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

5 In ΔABC, let D be a point on side BC. Suppose the incircle ω_1 of ΔABD touches sides AB and AD at E, F respectively, and the incircle ω_2 of ΔACD touches sides AD and AC at F, G respectively. Suppose the segment EG intersects ω_1 and ω_2 again at P and Q respectively. Show that line AD, tangent of ω_1 at P and tangent of ω_2 at Q are concurrent.

6 For a sequence with some ones and zeros, we count the number of continuous runs of equal digits in it. (For example the sequence 011001010 has 7 continuous runs: $0, 11, 00, 1, 0, 1, 0$.) Find the sum of the number of all continuous runs for all possible sequences with 2019 ones and 2019 zeros.

Test 2 October 26, 2019

1 Let ΔABC be an acute triangle with incenter I and orthocenter H. AI meets the circumcircle of ΔABC again at M. Suppose the length IM is exactly the circumradius of ΔABC. Show that $AH \geq AI$.

2 Suppose there are 2019 distinct points in a plane and the distances between pairs of them attain k different values. Prove that k is at least 44.

3 Two circles Γ and Ω intersect at two distinct points A and B. Let P be a point on Γ. The tangent at P to Γ meets Ω at the points C and D, where D lies between P and C, and $ABCD$ is a convex quadrilateral. The lines CA and CB meet Γ again at E and F respectively. The lines DA and DB meet Γ again at S and T respectively. Suppose the points P, E, S, F, B, T, A lie on Γ in this order. Prove that PC, ET, SF are parallel.

4 Find all real-valued functions f defined on the set of real numbers such that

$$f(f(x)+y)+f(x+f(y))=2f(xf(y))$$

for any real numbers x and y.