Romania National Olympiad 2017

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–	Grade 9
1	Prove that the line joining the centroid and the incenter of a non-isosceles triangle is perpendicular to the base if and only if the sum of the other two sides is thrice the base.
2	Let be a square $ABCD$, a point E on AB, a point N on CD, points F, M on BC, name P the intersection of AN with DE, and name Q the intersection of AM with EF. If the triangles AMN and DEF are equilateral, prove that $PQ = FM$.
3	Let be two natural numbers n and a. **a)** Prove that there exists an n-tuplet of natural numbers $(a_1, a_2, \ldots, a_n)$ that satisfy the following equality. $$1 + \frac{1}{a} = \prod_{i=1}^{n} \left(1 + \frac{1}{a_i}\right)$$ **b)** Show that there exist only finitely such n-tuplets.
4	Let be two natural numbers $b > a > 0$ and a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ having the following property. $$f\left(x^2 + ay\right) \geq f\left(x^2 + by\right), \quad \forall x, y \in \mathbb{R}$$ **a)** Show that $f(s) \leq f(0) \leq f(t)$, for any real numbers $s < 0 < t$. **b)** Prove that f is constant on the interval $(0, \infty)$. **c)** Give an example of a non-monotone such function.
–	Grade 10
1	Solve in the set of real numbers the equation $a^{[x]} + \log_a\{x\} = x$, where a is a real number from the interval $(0, 1)$. $[]$ and $\{\}$ *denote the floor, respectively, the fractional part.*
2	A function $f : \mathbb{Q}_{>0} \longrightarrow \mathbb{Q}$ has the following property: $$f(xy) = f(x) + f(y), \quad x, y \in \mathbb{Q}_{>0}$$ **a)** Demonstrate that there are no injective functions with this property. **b)** Do exist surjective functions having this property?

3 $\sin\frac{\pi}{4n} \geq \frac{\sqrt{2}}{2n}, \quad \forall n\in\mathbb{N}$

4 Find the number of functions $A\stackrel{f}{\longrightarrow} A$ for which there exist two functions $A\stackrel{g}{\longrightarrow} B\stackrel{h}{\longrightarrow} A$ having the properties that $g\circ h=\text{id.}$ and $h\circ g=f$, where B and A are two finite sets.

– Grade 11

1 Let be a surjective function $f:\mathbb{R}\longrightarrow\mathbb{R}$ that has the property that if the sequence $\left(f\left(x_n\right)\right)_{n\ge 1}$ is convergent, then the sequence $\left(x_n\right)_{n\ge 1}$ is convergent. Prove that it is continuous.

2 Let be two natural numbers $n\ge 2, k,$ and $k\quad n\times n$ symmetric real matrices $A_1,A_2,\ldots ,A_k.$ Then, the following relations are equivalent:

$1)\quad \left| \sum_{i=1}^k A_i^2\right| =0$

$2)\quad \left| \sum_{i=1}^k A_iB_i\right| =0,\quad \forall B_1,B_2,\ldots ,B_k\in\mathcal{M}_n\left(\mathbb{R}\right)$

$||$ *denotes the determinant.*

3 Let be a natural number $n\ge 2$ and two $n\times n$ complex matrices A,B that satisfy $(AB)^3=O_n.$ Does this imply that $(BA)^3=O_n?$

4 Let be a function f of class $\mathcal{C}^1[a,b]$ whose derivative is positive. Prove that there exists a real number $c\in (a,b)$ such that

$$f(f(b))-f(f(a))=(f'(c))^2(b-a).$$

– Grade 12

1 **a)** Let be a continuous function $f:\mathbb{R}_{\ge 0}\longrightarrow\mathbb{R}_{>0}.$ Show that there exists a natural number n_0 and a sequence of positive real numbers $\left(x_n\right)_{n>n_0}$ that satisfy the following relation.

$$n\int_0^{x_n} f(t)dt=1,\quad n_0<\forall n\in\mathbb{N}$$

b) Prove that the sequence $\left(nx_n\right)_{n> n_0}$ is convergent and find its limit.

2 Let be a natural number n and $2n$ real numbers $b_1,b_2,\ldots ,b_n,a_1<a_2<\cdots <a_n.$ Show that

a) if $b_1,b_2,\ldots ,b_n>0,$ then there exists a polynomial $f\in\mathbb{R} [X]$ irreducible in $\mathbb{R} [X]$ such that

$$f\left(a_i\right) =b_i,\quad\forall i\in\{ 1,2,\ldots ,n\} .$$

b) there exists a polynom $g \in \mathbb{R}[X]$ of degree at least 1 which has only real roots and such that

$$g\left(a_i\right) = b_i, \quad \forall i \in \{1, 2, \ldots, n\}.$$

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3 Let G be a finite group with the following property:

If f is an automorphism of G, then there exists $m \in \mathbb{N}^\star$, so that $f(x) = x^m$ for all $x \in G$.

Prove that G is commutative.

Marian Andronache

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4 A function $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ has the property that $\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)dt = 1$.

a) Give an example of what f could be if it's continuous and $f/$id. doesn't have a limit at ∞.
b) Prove that if f is nondecreasing then $f/$id. has a limit at ∞, and determine it.
