2nd Bay Area Mathematical Olympiad 2000

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1 Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22+15$ represents the number 37 in the desired way.)

2 Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F, let k_2 be the circle passing through points B, E, and D, and let k_3 be the circle passing through C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

3 Let $x_1, x_2, ..., x_n$ be positive numbers, with $n \ge 2$. Prove that

$$\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)...\left(x_n+\frac{1}{x_n}\right) \ge \left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)...\left(x_{n-1}+\frac{1}{x_n}\right)\left(x_n+\frac{1}{x_1}\right)$$

4 Prove that there exists a set S of 3^{1000} points in the plane such that for each point P in S, there are at least 2000 points in S whose distance to P is exactly 1 inch.

5 Alice plays the following game of solitaire on a 20×20 chessboard.
She begins by placing 100 pennies, 100 nickels, 100 dimes, and 100 quarters on the board so that each of the 400 squares contains exactly one coin. She then chooses 59 of these coins and removes them from the board.
After that, she removes coins, one at a time, subject to the following rules:
- A penny may be removed only if there are four squares of the board adjacent to its square (up, down, left, and right) that are vacant (do not contain coins). Squares off the board do not count towards this four: for example, a non-corner square bordering the edge of the board has three adjacent squares, so a penny in such a square cannot be removed under this rule, even if all three adjacent squares are vacant.
- A nickel may be removed only if there are at least three vacant squares adjacent to its square. (And again, off the board squares do not count.)
- A dime may be removed only if there are at least two vacant squares adjacent to its square (off the board squares do not count).
- A quarter may be removed only if there is at least one vacant square adjacent to its square (off the board squares do not count).

Alice wins if she eventually succeeds in removing all the coins. Prove that it is impossiblefor her to win.