3rd Bay Area Mathematical Olympiad 2001

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1 Each vertex of a regular 17-gon is colored red, blue, or green in such a way that no two adjacent vertices have the same color. Call a triangle multicolored if its vertices are colored red, blue, and green, in some order. Prove that the 17-gon can be cut along nonintersecting diagonals to form at least two multicolored triangles.
(A diagonal of a polygon is a a line segment connecting two nonadjacent vertices. Diagonals are called nonintersecting if each pair of them either intersect in a vertex or do not intersect at all.)

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2 Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y, and Z are collinear (lie on the same line).

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3 Let $f(n)$ be a function satisfying the following three conditions for all positive integers n:
(a) $f(n)$ is a positive integer,
(b) $f(n+1) > f(n)$,
(c) $f(f(n)) = 3n$.
Find $f(2001)$.

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4 A kingdom consists of 12 cities located on a one-way circular road. A magician comes on the 13th of every month to cast spells. He starts at the city which was the 5th down the road from the one that he started at during the last month (for example, if the cities are numbered 112 clockwise, and the direction of travel is clockwise, and he started at city#9 last month, he will start at city#2 this month). At each city that he visits, the magician casts a spell if the city is not already under the spell, and then moves on to the next city. If he arrives at a city which is already under the spell, then he removes the spell from this city, and leaves the kingdom until the next month. Last Thanksgiving the capital city was free of the spell. Prove that it will be free of the spell this Thanksgiving as well.

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5 For each positive integer n, let a_n be the number of permutations τ of $\{1, 2, ..., n\}$ such that $\tau(\tau(\tau(x))) = x$ for $x = 1, 2, ..., n$. The first few values are $a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 9.$
Prove that 3^{334} divides a_{2001}.
(A permutation of $\{1, 2, ..., n\}$ is a rearrangement of the numbers $\{1, 2, ..., n\}$ or equivalently, a one-to-one and
onto function from $\{1, 2, ..., n\}$ to $\{1, 2, ..., n\}$. For example, one permutation of $\{1, 2, 3\}$ is the rearrangement $\{2, 1, 3\}$, which is equivalent to the function $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by

$\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3.$)