Problems from the 2018 BAMO-8 and BAMO-12 exams

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A Twenty-five people of different heights stand in a 5×5 grid of squares, with one person in each square. We know that each row has a shortest person, suppose Ana is the tallest of these five people. Similarly, we know that each column has a tallest person, suppose Bev is the shortest of these five people.
Assuming Ana and Bev are not the same person, who is taller: Ana or Bev?
Prove that your answer is always correct.

B A square with sides of length 1 cm is given. There are many different ways to cut the square into four rectangles.
Let S be the sum of the four rectangles perimeters. Describe all possible values of S with justification.

C/1 An integer c is *square-friendly* if it has the following property:
For every integer m, the number $m^2 + 18m + c$ is a perfect square.
(A perfect square is a number of the form n^2, where n is an integer. For example, $49 = 7^2$ is a perfect square while 46 is not a perfect square. Further, as an example, 6 is not *square-friendly* because for $m = 2$, we have $(2)2 + (18)(2) + 6 = 46$, and 46 is not a perfect square.)
In fact, exactly one square-friendly integer exists. Show that this is the case by doing the following:
(a) Find a square-friendly integer, and prove that it is square-friendly.
(b) Prove that there cannot be two different square-friendly integers.

D/2 Let points P_1, P_2, P_3, and P_4 be arranged around a circle in that order. (One possible example is drawn in Diagram 1.) Next draw a line through P_4 parallel to P_1P_2, intersecting the circle again at P_5. (If the line happens to be tangent to the circle, we simply take $P_5 = P_4$, as in Diagram 2. In other words, we consider the second intersection to be the point of tangency again.) Repeat this process twice more, drawing a line through P_5 parallel to P_2P_3, intersecting the circle again at P_6, and finally drawing a line through P_6 parallel to P_3P_4, intersecting the circle again at P_7. Prove that P_7 is the same point as P_1.
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`png`

E/3 Suppose that 2002 numbers, each equal to 1 or -1, are written around a circle. For every two adjacent numbers, their product is taken; it turns out that the sum of all 2002 such products is negative. Prove that the sum of the original numbers has absolute value less than or equal to 1000. (The absolute value of x is usually denoted by $|x|$. It is equal to x if $x \geq 0$, and to $-x$ if $x < 0$. For example, $|6| = 6$, $|0| = 0$, and $|-7| = 7$.)

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4 (a) Find two quadruples of positive integers (a, b, c, n), each with a different value of n greater than 3, such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = n$.
(b) Show that if a, b, c are nonzero integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer, then abc is a perfect cube.
(A perfect cube is a number of the form n^3, where n is an integer.)

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5 To *dissect* a polygon means to divide it into several regions by cutting along finitely many line segments. For example, the diagram below shows a dissection of a hexagon into two triangles and two quadrilaterals:
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An *integer-ratio* right triangle is a right triangle whose side lengths are in an integer ratio. For example, a triangle with sides $3, 4, 5$ is an *integer-ratio* right triangle, and so is a triangle with sides $\frac{5}{2}\sqrt{3}, 6\sqrt{3}, \frac{13}{2}\sqrt{3}$. On the other hand, the right triangle with sides $\sqrt{2}, \sqrt{5}, \sqrt{7}$ is not an *integer-ratio* right triangle. Determine, with proof, all integers n for which it is possible to completely *dissect* a regular n-sided polygon into *integer-ratio* right triangles.
