5th Bay Area Mathematical Olympiad 2003

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1 An integer is a perfect number if and only if it is equal to the sum of all of its divisors except itself.
For example, 28 is a perfect number since $28 = 1 + 2 + 4 + 7 + 14$.
Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdot ... \cdot n$, where n is a positive integer.
An integer is a factorial if and only if it is equal to $n!$ for some positive integer n.
For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.
Find all perfect numbers greater than 1 that are also factorials.

2 Five mathematicians find a bag of 100 gold coins in a room. They agree to split up the coins according to the following plan:
The oldest person in the room proposes a division of the coins among those present. (No coin may be split.) Then all present, including the proposer, vote on the proposal.
If at least 50% of those present vote in favor of the proposal, the coins are distributed accordingly and everyone goes home. (In particular, a proposal wins on a tie vote.)
If fewer than 50% of those present vote in favor of the proposal, the proposer must leave the room, receiving no coins. Then the process is repeated: the oldest person remaining proposes a division,
and so on.
There is no communication or discussion of any kind allowed, other than what is needed for the proposer to state his or her proposal, and the voters to cast their vote.
Assume that each person is equally intelligent and each behaves optimally to maximize his or her share.
How much will each person get?

3 A lattice point is a point (x, y) with both x and y integers. Find, with proof, the smallest n such that every set of n lattice points contains three points that are the vertices of a triangle with integer area. (The triangle may be degenerate, in other words, the three points may lie on a straight line and hence form a triangle with area zero.)

4 An integer $n > 1$ has the following property: for every (positive) divisor d of $n, d + 1$ is a divisor of $n + 1$.
Prove that n is prime.

5 Let $ABCD$ be a square, and let E be an internal point on side AD. Let F be the foot of the perpendicular from B to CE. Suppose G is a point such that $BG = FG$, and the line through

G parallel to BC passes through the midpoint of EF. Prove that $AC < 2 \cdot FG$.