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Art of Problem Solving

2012, Bay Area Mathematical Olympiad 8/12

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– BAMO-8

1 Hugo places a chess piece on the top left square of a 20×20 chessboard and makes 10 moves with it. On each of these 10 moves, he moves the piece either one square horizontally (left or right) or one square vertically (up or down). After the last move, he draws an X on the square that the piece occupies. When Hugo plays the game over and over again, what is the largest possible number of squares that could eventually be marked with an X? Prove that your answer is correct.

2 Answer the following two questions and justify your answers:
(a) What is the last digit of the sum $1^{2012}+2^{2012}+3^{2012}+4^{2012}+5^{2012}$?
(b) What is the last digit of the sum $1^{2012}+2^{2012}+3^{2012}+4^{2012}+...+2011^{2012}+2012^{2012}$?

3 Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:
-No two arrows point at the same dot.
-Now arrow can extend right or left by more than 1006 positions.
`https://cdn.artofproblemsolving.com/attachments/7/6/47abf37771176fce21bce057edf0429d0181f png`
Show that at most 2012 dots in the lower row could have no arrow pointing to them.

4 Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane; but she can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler; with it, she found the midpoint M of side CD, and she extended CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M, and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.

– BAMO-12

1 Same as BAMO-8#2

2 Same as BAMO-8#3

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3 Let $x_1, x_2, ..., x_k$ be a sequence of integers. A rearrangement of this sequence (the numbers in the sequence listed in some other order) is called a **scramble** if no number in the new sequence is equal to the number originally in its location. For example, if the original sequence is $1, 3, 3, 5$ then $3, 5, 1, 3$ is a scramble, but $3, 3, 1, 5$ is not.

A rearrangement is called a **two-two** if exactly two of the numbers in the new sequence are each exactly two more than the numbers that originally occupied those locations. For example, $3, 5, 1, 3$ is a two-two of the sequence $1, 3, 3, 5$ (the first two values 3 and 5 of the new sequence are exactly two more than their original values 1 and 3).

Let $n \geq 2$. Prove that the number of scrambles of $1, 1, 2, 3, ..., n-1, n$ is equal to the number of two-twos of $1, 2, 3, ..., n, n+1$.

(Notice that both sequences have $n+1$ numbers, but the first one contains two 1s.)

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4 Given a segment AB in the plane, choose on it a point M different from A and B. Two equilateral triangles $\triangle AMC$ and $\triangle BMD$ in the plane are constructed on the same side of segment AB. The circumcircles of the two triangles intersect in point M and another point N. (The **circumcircle** of a triangle is the circle that passes through all three of its vertices.)
(a) Prove that lines AD and BC pass through point N.
(b) Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

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5 Find all nonzero polynomials $P(x)$ with integers coefficients that satisfy the following property: whenever a and b are relatively prime integers, then $P(a)$ and $P(b)$ are relatively prime as well. Prove that your answer is correct. (Two integers are **relatively prime** if they have no common prime factors. For example, -70 and 99 are relatively prime, while -70 and 15 are not relatively prime.)
