**6th Bay Area Mathematical Olympiad 2004**

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**1** A tiling of the plane with polygons consists of placing the polygons in the plane so that interiors of polygons do not overlap, each vertex of one polygon coincides with a vertex of another polygon, and no point of the plane is left uncovered. A unit polygon is a polygon with all sides of length one. It is quite easy to tile the plane with infinitely many unit squares. Likewise, it is easy to tile the plane with infinitely many unit equilateral triangles.
(a) Prove that there is a tiling of the plane with infinitely many unit squares and infinitely many unit equilateral triangles in the same tiling.
(b) Prove that it is impossible to find a tiling of the plane with infinitely many unit squares and finitely many (and at least one) unit equilateral triangles in the same tiling.

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**2** A given line passes through the center $O$ of a circle. The line intersects the circle at points $A$ and $B$. Point $P$ lies in the exterior of the circle and does not lie on the line $AB$. Using only an unmarked straightedge, construct a line through $P$, perpendicular to the line $AB$. Give complete instructions for the construction and prove that it works.

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**3** NASA has proposed populating Mars with $2,004$ settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing $N$ tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of $N$ that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

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**4** Suppose one is given $n$ real numbers, not all zero, but such that their sum is zero.
Prove that one can label these numbers $a_1, a_2, ..., a_n$ in such a manner that $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1 < 0$.

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**5** Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.
1. $f(0) = 2004$.
2. If $x$ is irrational, then $f(x)$ is also irrational.

(Notes: Apolynomial is monic if its highest degree term has coefficient $1$. Thus, $f(x) = x^4 - 5x^3 - 4x + 7$ is an example of a monic polynomial with integer coefficients.
A number $x$ is rational if it can be written as a fraction of two integers. A number $x$ is irrational if it is a real number which cannot be written as a fraction of two integers. For example, $2/5$ and $-9$ are rational, while $\sqrt{2}$ and $\pi$ are well known to be irrational.)

 