7th Bay Area Mathematical Olympiad 2005

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1 An integer is called *formidable* if it can be written as a sum of distinct powers of 4, and *successful* if it can be written as a sum of distinct powers of 6. Can 2005 be written as a sum of a *formidable* number and a *successful* number? Prove your answer.

2 Prove that if two medians in a triangle are equal in length, then the triangle is isosceles.

(Note: A median in a triangle is a segment which connects a vertex of the triangle to the midpoint of the opposite side of the triangle.)

3 Let $n \geq 12$ be an integer, and let $P_1, P_2, ...P_n, Q$ be distinct points in a plane. Prove that for some i, at least $\frac{n}{6} - 1$ of the distances $P_1P_i, P_2P_i, ...P_{i-1}P_i, P_{i+1}P_i, ...P_nP_i$ are less than P_iQ.

4 There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

5 Let D be a dodecahedron which can be inscribed in a sphere with radius R. Let I be an icosahedron which can also be inscribed in a sphere of radius R. Which has the greater volume, and why?

Note: A regular *polyhedron* is a geometric solid, all of whose faces are congruent regular polygons, in which the same number of polygons meet at each vertex. A regular dodecahedron is a polyhedron with 12 faces which are regular pentagons and a regular icosahedron is a polyhedron with 20 faces which are equilateral triangles. A polyhedron is inscribed in a sphere if all of its vertices lie on the surface of the sphere.

The illustration below shows a dodecahdron and an icosahedron, not necessarily to scale.
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