8th Bay Area Mathematical Olympiad 2006

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1 All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:
(a) Every student must move to a new chair.
(b) A student can only move to an adjacent chair in the same row or to an adjacent chair in the same
column. In other words, each student can move only one chair horizontally or vertically.
(Note that the rules above allow two students in adjacent chairs to exchange places.)
Show that this procedure can be done if n is even, and cannot be done if n is odd.

2 Since $24 = 3+5+7+9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.
(a) Can 2005 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.
(b) Can 2006 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

3 In triangle ABC, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of triangle ABC if and only if P is the centroid of triangle $A_1B_1C_1$.

Note: A median in a triangle is a segment connecting a vertex of the triangle with the midpoint of the opposite side. The centroid of a triangle is the intersection point of the three medians of the triangle. The centroid of a triangle is also known by the names center of mass and medicenter of the triangle.

4 Suppose that n squares of an infinite square grid are colored grey, and the rest are colored white. At each step, a new grid of squares is obtained based on the previous one, as follows. For each location in the grid, examine that square, the square immediately above, and the square immediately to the right.
If there are two or three grey squares among these three, then in the next grid, color that location grey, otherwise, color it white. Prove that after at most n steps all the squares in the grid will be white.
Below is an example with $n = 4$. The first grid shows the initial configuration, and the second grid shows the configuration after one step.

https://cdn.artofproblemsolving.com/attachments/1/a/87f7e3892cdb45fb3529127234aae2cea0874
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5 We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.