9th Bay Area Mathematical Olympiad 2007

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–	February 27, 2007

1	A 15-inch-long stick has four marks on it, dividing it into five segments of length $1, 2, 3, 4$, and 5 inches (although not neccessarily in that order) to make a ruler. Here is an example.
	`https://cdn.artofproblemsolving.com/attachments/0/e/065d42b36083453f3586970125bedbc804b8a`
	`png`
	Using this ruler, you could measure 8 inches (between the marks B and D) and 11 inches (between the end of the ruler at A and the mark at E), but theres no way you could measure 12 inches.
	Prove that it is impossible to place the four marks on the stick such that the five segments have length $1, 2, 3, 4$, and 5 inches, and such that every integer distance from 1 inch through 15 inches could be measured.

2	The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.
3	In $\triangle\ ABC, D$ and E are two points on segment BC such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle\ ABC$ is isosceles
4	Let N be the number of ordered pairs (x, y) of integers such that $x^2 + xy + y^2 \leq 2007$. Remember, integers may be positive, negative, or zero! (a) Prove that N is odd. (b) Prove that N is not divisible by 3.
5	Two sequences of positive integers, $x_1, x_2, x_3, ...$ and $y_1, y_2, y_3, ..$ are given, such that $\frac{y_{n+1}}{x_{n+1}} > \frac{y_n}{x_n}$ for each $n \geq 1$. Prove that there are infinitely many values of n such that $y_n > \sqrt{n}$.