2008, Bay Area Mathematical Olympiad 8/12

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–	BAMO-8
1	Call a year *ultra-even* if all of its digits are even. Thus $2000, 2002, 2004, 2006$, and 2008 are all *ultra-even* years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an *ultra-even* year because of the 9, and 2010 is not an ultra-even year because of the 1. (a) In the years between the years 1 and 10000, what is the longest possible gap between two *ultra-even* years? Give an example of two ultra-even years that far apart with no *ultra-even* years between them. Justify your answer. (b) What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two *ultra-even* years? Again, give an example, and justify your answer.
2	Consider a 7×7 chessboard that starts out with all the squares white. We start painting squares black, one at a time, according to the rule that after painting the first square, each newly painted square must be adjacent along a side to only the square just previously painted. The final figure painted will be a connected snake of squares. (a) Show that it is possible to paint 31 squares. (b) Show that it is possible to paint 32 squares. (c) Show that it is possible to paint 33 squares.
3	A triangle is constructed with the lengths of the sides chosen from the set $\{2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}$. Show that this triangle must be isosceles. (A triangle is isosceles if it has at least two sides the same length.)

4	Determine the greatest number of figures congruent to `https://cdn.artofproblemsolving.com/attachments/c/6/343f9197bcebf6794460ed1a74ba83ec18a377.png` that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares. The figures can be rotated and flipped over. For example, the picture below shows that at least 3 such figures can be placed in a 4×4 grid.
	`https://cdn.artofproblemsolving.com/attachments/1/e/d38fc34b650a1333742bb206c29985c94146a`
	`png`

–	BAMO-12
1	Same as BAMO-8#3

2 Same as BAMO-8#4

3 N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a Californian team Alcatraz is the unique Californian champion (Alcatraz has won more games against Californian teams than any other team from California). However, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation!

What is the smallest possible value for N?

4 A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BA_1} + \frac{CD}{CC_1} = 1.$$

5 A positive integer N is called stable if it is possible to split the set of all positive divisors of N (including 1 and N) into two subsets that have no elements in common, which have the same sum. For example, 6 is stable, because $1+2+3=6$, but 10 is not stable. Is $2^{2008} \cdot 2008$ stable?