2010, Bay Area Mathematical Olympiad 8/12

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– BAMO-8

1 We write $\{a, b, c\}$ for the set of three different positive integers a, b, and c. By choosing some or all of the numbers a, b and c, we can form seven nonempty subsets of $\{a, b, c\}$. We can then calculate the sum of the elements of each subset. For example, for the set $\{4, 7, 42\}$ we will find sums of $4, 7, 42, 11, 46, 49$, and 53 for its seven subsets. Since $7, 11$, and 53 are prime, the set $\{4, 7, 42\}$ has exactly three subsets whose sums are prime. (Recall that prime numbers are numbers with exactly two different factors, 1 and themselves. In particular, the number 1 is not prime.)

What is the largest possible number of subsets with prime sums that a set of three different positive integers can have? Give an example of a set $\{a, b, c\}$ that has that number of subsets with prime sums, and explain why no other three-element set could have more.

2 A clue k digits, sum is n gives a number k and the sum of k distinct, nonzero digits. An answer for that clue consists of k digits with sum n. For example, the clue Three digits, sum is 23 has only one answer. $6, 8, 9$. The clue Three digits, sum is 8 has two answers: $1, 3, 4$ and $1, 2, 5$.
If the clue Four digits, sum is n has the largest number of answers for any four-digit clue, then what is the value of n? How many answers does this clue have? Explain why no other four-digit clue can have more answers.

3 Suppose a, b, c are real numbers such that $a + b \geq 0, b + c \geq 0$, and $c + a \geq 0$.
Prove that $a + b + c \geq \frac{|a|+|b|+|c|}{3}$.

(Note: $|x|$ is called the absolute value of x and is defined as follows.
If $x \geq 0$ then $|x| = x$, and if $x < 0$ then $|x| = -x$. For example, $|6| = 6, |0| = 0$ and $|-6| = 6$.)

4 Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red. Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

All the rooks are still on unpainted squares.
The rooks are still not attacking each other (no two are in the same row or same column).
At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

–	BAMO-12
1	Same as BAMO-8#3
2	Same as BAMO-8#4
3	All vertices of a polygon P lie at points with integer coordinates in the plane, and all sides of P have integer lengths. Prove that the perimeter of P must be an even number.
4	Acute triangle ABC has $\angle BAC < 45^\circ$. Point D lies in the interior of triangle ABC so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.
5	Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that $$\frac{1}{\sqrt{\frac{1}{2}+a+ab+abc}}+\frac{1}{\sqrt{\frac{1}{2}+b+bc+bcd}}+\frac{1}{\sqrt{\frac{1}{2}+c+cd+cda}}+\frac{1}{\sqrt{\frac{1}{2}+d+da+dab}}\geq\sqrt{2}$$