Middle European Mathematical Olympiad 2019

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by Tintarn, XbenX

– Individual Competition

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1 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for any two real numbers x,y holds

$$f(xf(y)+2y)=f(xy)+xf(y)+f(f(y)).$$

Proposed by Patrik Bak, Slovakia

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2 Let $n\geq 3$ be an integer. We say that a vertex $A_i (1\leq i\leq n)$ of a convex polygon $A_1A_2\dots A_n$ is *Bohemian* if its reflection with respect to the midpoint of $A_{i-1}A_{i+1}$ (with $A_0=A_n$ and $A_1=A_{n+1}$) lies inside or on the boundary of the polygon $A_1A_2\dots A_n$. Determine the smallest possible number of Bohemian vertices a convex n-gon can have (depending on n).

Proposed by Dominik Burek, Poland

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3 Let ABC be an acute-angled triangle with $AC>BC$ and circumcircle ω. Suppose that P is a point on ω such that $AP=AC$ and that P is an interior point on the shorter arc BC of ω. Let Q be the intersection point of the lines AP and BC. Furthermore, suppose that R is a point on ω such that $QA=QR$ and R is an interior point of the shorter arc AC of ω. Finally, let S be the point of intersection of the line BC with the perpendicular bisector of the side AB.
Prove that the points P,Q,R and S are concyclic.

Proposed by Patrik Bak, Slovakia

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4 Determine the smallest positive integer n for which the following statement holds true: From any n consecutive integers one can select a non-empty set of consecutive integers such that their sum is divisible by 2019.

Proposed by Kartal Nagy, Hungary

– Team Competition

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1 Determine the smallest and the greatest possible values of the expression

$$\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)$$

provided a,b and c are non-negative real numbers satisfying $ab+bc+ca=1$.

Proposed by Walther Janous, Austria

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2 Let α be a real number. Determine all polynomials P with real coefficients such that

$$P(2x+\alpha) \leq (x^{20}+x^{19})P(x)$$

holds for all real numbers x.

Proposed by Walther Janous, Austria

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3 There are n boys and n girls in a school class, where n is a positive integer. The heights of all the children in this class are distinct. Every girl determines the number of boys that are taller than her, subtracts the number of girls that are taller than her, and writes the result on a piece of paper. Every boy determines the number of girls that are shorter than him, subtracts the number of boys that are shorter than him, and writes the result on a piece of paper.
Prove that the numbers written down by the girls are the same as the numbers written down by the boys (up to a permutation).

Proposed by Stephan Wagner, Austria

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4 Prove that every integer from 1 to 2019 can be represented as an arithmetic expression consisting of up to 17 symbols 2 and an arbitrary number of additions, subtractions, multiplications, divisions and brackets. The 2's may not be used for any other operation, for example, to form multidigit numbers (such as 222) or powers (such as 2^2).

Valid examples:

$$\left((2\times 2+2)\times 2-\frac{2}{2}\right)\times 2=22 \quad , \quad (2\times 2\times 2-2)\times\left(2\times 2+\frac{2+2+2}{2}\right)=42$$

Proposed by Stephan Wagner, Austria

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5 Let ABC be an acute-angled triangle such that $AB<AC$. Let D be the point of intersection of the perpendicular bisector of the side BC with the side AC. Let P be a point on the shorter arc AC of the circumcircle of the triangle ABC such that $DP \parallel BC$. Finally, let M be the midpoint of the side AB. Prove that $\angle APD = \angle MPB$.

Proposed by Dominik Burek, Poland

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6 Let ABC be a right-angled triangle with the right angle at B and circumcircle c. Denote by D the midpoint of the shorter arc AB of c. Let P be the point on the side AB such that $CP=CD$ and let X and Y be two distinct points on c satisfying $AX=AY=PD$. Prove that X, Y and P are collinear.

Proposed by Dominik Burek, Poland

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7 Let a, b and c be positive integers satisfying $a < b < c < a + b$. Prove that $c(a-1) + b$ does not divide $c(b-1) + a$.

Proposed by Dominik Burek, Poland

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8 Let N be a positive integer such that the sum of the squares of all positive divisors of N is equal to the product $N(N+3)$. Prove that there exist two indices i and j such that $N = F_iF_j$ where $(F_i)_{n=1}^{\infty}$ is the Fibonacci sequence defined as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Proposed by Alain Rossier, Switzerland
