Dutch IMO Team Selection Test 2018

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–	Day 1
1	Suppose a grid with $2m$ rows and $2n$ columns is given, where m and n are positive integers. You may place one pawn on any square of this grid, except the bottom left one or the top right one. After placing the pawn, a snail wants to undertake a journey on the grid. Starting from the bottom left square, it wants to visit every square exactly once, except the one with the pawn on it, which the snail wants to avoid. Moreover, it wants to fi nish in the top right square. It can only move horizontally or vertically on the grid. On which squares can you put the pawn for the snail to be able to finish its journey?
2	Suppose a triangle $\triangle ABC$ with $\angle C = 90^o$ is given. Let D be the midpoint of AC, and let E be the foot of the altitude through C on BD. Show that the tangent in C of the circumcircle of $\triangle AEC$ is perpendicular to AB.
3	Let $n \ge 0$ be an integer. A sequence $a_0, a_1, a_2, ...$ of integers is de fined as follows: we have $a_0 = n$ and for $k \ge 1, a_k$ is the smallest integer greater than a_{k-1} for which $a_k + a_{k-1}$ is the square of an integer. Prove that there are exactly $\lfloor \sqrt{2n} \rfloor$ positive integers that cannot be written in the form $a_k - a_\ell$ with $k > \ell \ge 0$.
4	Let A be a set of functions $f : R \to R$. For all $f_1, f_2 \in A$ there exists a $f_3 \in A$ such that $f_1(f_2(y) - x) + 2x = f_3(x + y)$ for all $x, y \in R$. Prove that for all $f \in A$, we have $f(x - f(x)) = 0$ for all $x \in R$.
–	Day 2
1	(a) If $c(a^3 + b^3) = a(b^3 + c^3) = b(c^3 + a^3)$ with a, b, c positive real numbers, does $a = b = c$ necessarily hold? (b) If $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$ with a, b, c positive real numbers, does $a = b = c$ necessarily hold?
2	Find all positive integers n, for which there exists a positive integer k such that for every positive divisor d of n, the number $d - k$ is also a (not necessarily positive) divisor of n.
3	Let ABC be an acute triangle, and let D be the foot of the altitude through A. On AD, there are distinct points E and F such that $\|AE\| = \|BE\|$ and $\|AF\| = \|CF\|$. A point$T \neq D$ satis es $\angle BTE = \angle CTF = 90^o$. Show that $\|TA\|^2 = \|TB\| \cdot \|TC\|$.

4 In the classroom of at least four students the following holds: no matter which four of them take seats around a round table, there is always someone who either knows both of his neighbours, or does not know either of his neighbours. Prove that it is possible to divide the students into two groups such that in one of them, all students know one another, and in the other, none of the students know each other.

(Note: if student A knows student B, then student B knows student A as well.)

– Day 3

1 A set of lines in the plan is called *nice* i f every line in the set intersects an odd number of other lines in the set.
Determine the smallest integer $k \geq 0$ having the following property:
for each 2018 distinct lines $\ell_1, \ell_2, ..., \ell_{2018}$ in the plane, there exist lines $\ell_{2018+1}, \ell_{2018+2}, ..., \ell_{2018+k}$ such that the lines $\ell_1, \ell_2, ..., \ell_{2018+k}$ are distinct and form a *nice* set.

2 Find all functions $f : R \to R$ such that $f(x^2) - f(y^2) \leq (f(x) + y)(x - f(y))$ for all $x, y \in R$.

3 Determine all pairs (a, b) of positive integers such that $(a + b)^3 - 2a^3 - 2b^3$ is a power of two.

4 In a non-isosceles triangle ABC the centre of the incircle is denoted by I. The other intersection point of the angle bisector of $\angle BAC$ and the circumcircle of $\triangle ABC$ is D. The line through I perpendicular to AD intersects BC in F. The midpoint of the circle arc BC on which A lies, is denoted by M. The other intersection point of the line MI and the circle through B, I and C, is denoted by N. Prove that FN is tangent to the circle through B, I and C.