Federal Competition For Advanced Students, Part 1, 2008

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by parmenides51

1 What is the remainder of the number $1\binom{2008}{0}+2\binom{2008}{1}+...+2009\binom{2008}{2008}$ when divided by 2008?

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2 Given $a \in R^+$ and an integer $n > 4$ determine all n-tuples $(x_1, ..., x_n)$ of positive real numbers that satisfy the following system of equations: $\begin{cases} x_1x_2(3a-2x_3)=a^3 \\ x_2x_3(3a-2x_4)=a^3 \\ ... \\ x_{n-2}x_{n-1}(3a-2x_n)=a^3 \\ x_{n-1}x_n(3a-2x_1)=a^3 \\ x_nx_1(3a-2x_2)=a^3 \end{cases}$

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3 Let $p > 1$ be a natural number. Consider the set F_p of all non-constant sequences of non-negative integers that satisfy the recursive relation $a_{n+1}=(p+1)a_n-pa_{n-1}$ for all $n > 0$. Show that there exists a sequence (a_n) in F_p with the property that for every other sequence (b_n) in F_p, the inequality $a_n \le b_n$ holds for all n.

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4 In a triangle ABC let E be the midpoint of the side AC and F the midpoint of the side BC. Let G be the foot of the perpendicular from C to AB. Show that $\triangle EFG$ is isosceles if and only if $\triangle ABC$ is isosceles.
