Regional Competition For Advanced Students 2009

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by FelixD

1 Find the largest interval $M \subseteq \mathbb{R}^+$, such that for all a, b, c, $d \in M$ the inequality

$$\sqrt{ab}+\sqrt{cd} \ge \sqrt{a+b}+\sqrt{c+d}$$

holds. Does the inequality

$$\sqrt{ab}+\sqrt{cd} \ge \sqrt{a+c}+\sqrt{b+d}$$

hold too for all $a, b, c, d \in M$?
($\mathbb{R}^+$ denotes the set of positive reals.)

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2 How many integer solutions $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ does the equation

$$2x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2=9$$

have?

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3 Let D, E, F be the feet of the altitudes wrt sides BC, CA, AB of acute-angled triangle $\triangle ABC$, respectively. In triangle $\triangle CFB$, let P be the foot of the altitude wrt side BC. Define Q and R wrt triangles $\triangle ADC$ and $\triangle BEA$ analogously. Prove that lines AP, BQ, CR don't intersect in one common point.

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4 Two infinite arithmetic progressions are called considerable different if the do not only differ by the absence of finitely many members at the beginning of one of the sequences.
How many pairwise considerable different non-constant arithmetic progressions of positive integers that contain an infinite non-constant geometric progression $(b_n)_{n\ge 0}$ with $b_2 = 40\cdot 2009$ are there?
