Federal Competition For Advanced Students, Part 2 2004

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by parmenides51, valerie

–	Day 1
1	Prove without using advanced (differential) calculus: (a) For any real numbers a,b,c,d it holds that $a^6+b^6+c^6+d^6-6abcd \geq -2$. When does equality hold? (b) For which natural numbers k does some inequality of the form $a^k+b^k+c^k+d^k-kabcd \geq M_k$ hold for all real a,b,c,d? For each such k,
2	Show that every set $\{p_1, p_2, \dots, p_k\}$ of prime numbers fulfils the following: The sum of all unit fractions (that are fractions of the type $\frac{1}{n}$), whose denominators are exactly the k given prime factors (but in arbitrary powers with exponents unequal zero), is an unit fraction again. How big is this sum if $\frac{1}{2004}$ is among this summands? Show that for every set $\{p_1, p_2, \dots, p_k\}$ containing k prime numbers ($k>2$) is the sum smaller than $\frac{1}{N}$ with $N=2\cdot 3^{k-2}(k-2)!$
3	A trapezoid $ABCD$ with perpendicular diagonals AC and BD is inscribed in a circle k. Let k_a and k_c respectively be the circles with diameters AB and CD. Compute the area of the region which is inside the circle k, but outside the circles k_a and k_c.
–	Day 2
4	Show that there is an infinite sequence $a_1, a_2, ...$ of natural numbers such that $a_1^2+a_2^2+...+a_N^2$ is a perfect square for all N. Give a recurrent formula for one such sequence.
5	Solve the following system of equations in real numbers: $\begin{cases} a^2 = \dfrac{\sqrt{bc}\sqrt[3]{bcd}}{(b+c)(b+c+d)} \\ b^2 = \dfrac{\sqrt{cd}\sqrt[3]{cda}}{(c+d)(c+d+a)} \\ c^2 = \dfrac{\sqrt{da}\sqrt[3]{dab}}{(d+a)(d+a+b)} \\ d^2 = \dfrac{\sqrt{ab}\sqrt[3]{abc}}{(a+b)(a+b+c)} \end{cases}$
6	Over the sides of an equilateral triangle with area 1 are triangles with the opposite angle 60° to each side drawn outside of the triangle. The new corners are P, Q and R. (and the new triangles APB, BQC and ARC)

1)What is the highest possible area of the triangle PQR?
2)What is the highest possible area of the triangle whose vertexes are the midpoints of the inscribed circles of the triangles APB, BQC and ARC?