Federal Competition For Advanced Students, Part 2 2009

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– Day 1

1 If $x, y, K, m \in N$, let us define:

$$a_m = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k\,twos}, A_{km}(x) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{x^{a_m}}}}}}}_{k\,twos}, B_k(y) = \underbrace{4^{4^{4^{\cdot^{\cdot^{\cdot^{4^y}}}}}}}_{m\,fours},$$

Determine all pairs (x, y) of non-negative integers, dependent on $k > 0$, such that $A_{km}(x) = B_k(y)$

2 (i) For positive integers $a < b$, let $M(a, b) = \frac{\Sigma_{k=a}^{b} \sqrt{k^2+3k+3}}{b-a+1}$.
Calculate $[M(a, b)]$
(ii) Calculate $N(a, b) = \frac{\Sigma_{k=a}^{b} [\sqrt{k^2+3k+3}]}{b-a+1}$.

3 Let P be the point in the interior of $\triangle ABC$. Let D be the intersection of the lines AP and BC and let A' be the point such that $\overrightarrow{AD} = \overrightarrow{DA'}$. The points B' and C' are defined in the similar way. Determine all points P for which the triangles $A'BC$, $AB'C$, and ABC' are congruent to $\triangle ABC$.

– Day 2

4 Let a be a positive integer. Consider the sequence (a_n) defined as $a_0 = a$
and $a_n = a_{n-1} + 40^{n!}$ for $n > 0$. Prove that the sequence (a_n) has infinitely
many numbers divisible by 2009.

5 Let $n > 1$ and for $1 \leq k \leq n$ let $p_k = p_k(a_1, a_2, ..., a_n)$ be the sum of the products of all possible combinations of k of the numbers $a_1, a_2, ..., a_n$. Furthermore let $P = P(a_1, a_2, ..., a_n)$ be the sum of all p_k with odd values of k less than or equal to n.

How many different values are taken by a_j if all the numbers $a_j (1 \leq j \leq n)$ and P are prime?

6 The quadrilateral PQRS whose vertices are the midpoints of the sides AB, BC,
CD, DA, respectively of a quadrilateral ABCD is called the midpoint quadrilateral
of ABCD.
Determine all circumscribed quadrilaterals whose mid-point quadrilaterals are
squares.
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