Belarusian National Olympiad 2019

www.artofproblemsolving.com/community/c939475

by Vlados021

– Category C

9.1 Is it true that for any nonzero rational numbers a and b one can find integers m and n such that the number $(am+b)^2+(a+nb)^2$ is an integer?

(M. Karpuk)

9.2 The rhombus $ABCD$ is given. Let E be one of the points of intersection of the circles Γ_B and Γ_C, where Γ_B is the circle centered at B and passing through C, and Γ_C is the circle centered at C and passing through B. The line ED intersects Γ_B at point F.
Find the value of angle $\angle AFB$.

(S. Mazanik)

9.3 Positive real numbers a and b satisfy the following conditions: the function $f(x)=x^3+ax^2+2bx-1$ has three different real roots, while the function $g(x)=2x^2+2bx+a$ doesn't have real roots.
Prove that $a-b>1$.

(V. Karamzin)

9.4 The sum of several (not necessarily different) positive integers not exceeding 10 is equal to S. Find all possible values of S such that these numbers can always be partitioned into two groups with the sum of the numbers in each group not exceeding 70.

(I. Voronovich)

9.5 For a positive integer n write down all its positive divisors in increasing order: $1=d_1<d_2<\ldots<d_k=n$.
Find all positive integers n divisible by 2019 such that $n=d_{19}\cdot d_{20}$.

(I. Gorodnin)

9.6 The point M is the midpoint of the side BC of triangle ABC. A circle is passing through B, is tangent to the line AM at M, and intersects the segment AB secondary at the point P.
Prove that the circle, passing through A, P, and the midpoint of the segment AM, is tangent to the line AC.

(A. Voidelevich)

9.7 Find all non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients such that $P(Q(x)^2) = P(x) \cdot Q(x)^2$.

(I. Voronovich)

9.8 Andrey and Sasha play the game, making moves alternate. On his turn, Andrey marks on the plane an arbitrary point that has not yet been marked. After that, Sasha colors this point in one of two colors: white and black. Sasha wins if after his move it is impossible to draw a line such that all white points lie in one half-plane, while all black points lie in another half-plane with respect to this line.
a) Prove that Andrey can make moves in such a way that Sasha will never win.
b) Suppose that Andrey can mark only integer points on the Cartesian plane. Can Sasha guarantee himself a win regardless of Andrey's moves?

(N. Naradzetski)

– Category B

10.1 The two lines with slopes 2 and $1/2$ pass through an arbitrary point T on the axis Oy and intersect the hyperbola $y = 1/x$ at two points.
a) Prove that these four points lie on a circle.
b) The point T runs through the entire y-axis. Find the locus of centers of such circles.

(I. Gorodnin)

10.2 A point P is chosen in the interior of the side BC of triangle ABC. The points D and C are symmetric to P with respect to the vertices B and C, respectively. The circumcircles of the triangles ABE and ACD intersect at the points A and X. The ray AB intersects the segment XD at the point C_1 and the ray AC intersects the segment XE at the point B_1.
Prove that the lines BC and B_1C_1 are parallel.

(A. Voidelevich)

10.3 The polynomial of seven variables

$$Q(x_1, x_2, \ldots, x_7) = (x_1 + x_2 + \ldots + x_7)^2 + 2(x_1^2 + x_2^2 + \ldots + x_7^2)$$

is represented as the sum of seven squares of the polynomials with nonnegative integer coefficients:

$$Q(x_1, \ldots, x_7) = P_1(x_1, \ldots, x_7)^2 + P_2(x_1, \ldots, x_7)^2 + \ldots + P_7(x_1, \ldots, x_7)^2.$$

Find all possible values of $P_1(1, 1, \ldots, 1)$.

(A. Yuran)

10.4 The sum of several (not necessarily different) real numbers from $[0,1]$ doesn't exceed S.
Find the maximum value of S such that it is always possible to partition these numbers into two groups with sums not greater than 9.

(I. Gorodnin)

10.5 Find all non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying the equality $P(Q(x)) = P(x)Q(x) - P(x)$.

(I. Voronovich)

10.6 The tangents to the circumcircle of the acute triangle ABC, passing through B and C, meet at point F. The points M, L, and N are the feet of perpendiculars from the vertex A to the lines FB, FC, and BC, respectively.
Prove the inequality $AM + AL \geq 2AN$.

(V. Karamzin)

10.7 The numbers $S_1 = 2^2, S_2 = 2^4, \ldots, S_n = 2^{2n}$ are given. A rectangle $OABC$ is constructed on the Cartesian plane according to these numbers. For this, starting from the point O the points $A_1, A_2, \ldots, A_n$ are consistently marked along the axis Ox, and the points $C_1, C_2, \ldots, C_n$ are consistently marked along the axis Oy in such a way that for all k from 1 to n the lengths of the segments $A_{k-1}A_k = x_k$ and $C_{k-1}C_k = y_k$ are positive integers (let $A_0 = C_0 = O$, $A_n = A$, and $C_n = C$) and $x_k \cdot y_k = S_k$.
a) Find the maximal possible value of the area of the rctangle $OABC$ and all pairs of sets $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ at which this maximal area is achieved.
b) Find the minimal possible value of the area of the rctangle $OABC$ and all pairs of sets $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ at which this minimal area is achieved.

(E. Manzhulina, B. Rublyov)

10.8 Call a polygon on a Cartesian plane to be*integer* if all its vertices are integer. A convex integer 14-gon is cut into integer parallelograms with areas not greater than C.
Find the minimal possible C.

(A. Yuran)

– Category A

11.1 **a)** Find all real numbers a such that the parabola $y = x^2 - a$ and the hyperbola $y = 1/x$ intersect each other in three different points.
b) Find the locus of centers of circumcircles of such triples of intersection points when a takes all possible values.

(I. Gorodnin)

11.2 The polynomial

$$Q(x_1, x_2, \ldots, x_4) = 4(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$$

is represented as the sum of squares of four polynomials of four variables with integer coefficients.
a) Find at least one such representation
b) Prove that for any such representation at least one of the four polynomials isidentically zero.

(A. Yuran)

11.3 The sum of several (not necessarily different) real numbers from $[0,1]$ doesn't exceed S.
Find the maximum value of S such that it is always possible to partition these numbers into two groups with sums $A \le 8$ and $B \le 4$.

(I. Gorodnin)

11.4 The altitudes CC_1 and BB_1 are drawn in the acute triangle ABC. The bisectors of angles $\angle BB_1C$ and $\angle CC_1B$ intersect the line BC at points D and E, respectively, and meet each other at point X.
Prove that the intersection points of circumcircles of the triangles BEX and CDX lie on the line AX.

(A. Voidelevich)

11.5 $n \ge 2$ positive integers are written on the blackboard. A move consists of three steps: 1) choose an arbitrary number a on the blackboard, 2) calculate the least common multiple N of all numbers written on the blackboard, and 3) replace a by N/a.
Prove that using such moves it is always possible to make all the numbers on the blackboard equal to 1.

(A. Naradzetski)

11.6 The diagonals of the inscribed quadrilateral $ABCD$ intersect at the point O. The points P, Q, R, and S are the feet of the perpendiculars from O to the sides AB, BC, CD, and DA, respectively.
Prove the inequality $BD \ge SP + QR$.

(A. Naradzetski)

11.7 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equality

$$f(f(x) + f(y)) = (x + y)f(x + y)$$

for all real x and y.

(B. Serankou)

11.8 At each node of the checkboard $n \times n$ board, a beetle sat. At midnight, each beetle crawled into the center of a cell. It turned out that the distance between any two beetles sitting in the adjacent (along the side) nodes didn't increase.
Prove that at least one beetle crawled into the center of a cell at the vertex of which it sat initially.

(A. Voidelevich)