Dutch Mathematical Olympiad 2008

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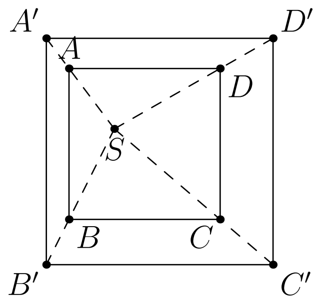
1 Suppose we have a square $ABCD$ and a point S in the interior of this square.
Under homothety with centre S and ratio of magnification $k > 1$, this square becomes another square $A'B'C'D'$.
Prove that the sum of the areas of the two quadrilaterals $A'ABB'$ and $C'CDD'$ are equal to the sum of the areas of the two quadrilaterals $B'BCC'$ and $D'DAA'$.



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2 Find all positive integers (m, n) such that $3 \cdot 2^n + 1 = m^2$.

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3 Suppose that we have a set S of 756 arbitrary integers between 1 and 2008 (1 and 2008 included).
Prove that there are two distinct integers a and b in S such that their sum $a + b$ is divisible by 8.

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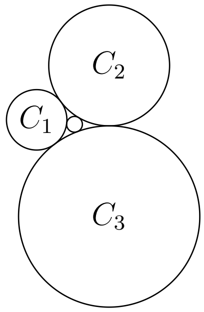
4 Three circles C_1, C_2, C_3, with radii $1, 2, 3$ respectively, are externally tangent.
In the area enclosed by these circles, there is a circle C_4 which is externally tangent to all three circles.
Find the radius of C_4.



5 We're playing a game with a sequence of 2008 non-negative integers.
A move consists of picking a integer b from that sequence, of which the neighbours a and c are positive. We then replace a, b and c by $a-1, b+7$ and $c-1$ respectively. It is not allowed to pick the first or the last integer in the sequence, since they only have one neighbour. If there is no integer left such that both of its neighbours are positive, then there is no move left, and the game ends.
Prove that the game always ends, regardless of the sequence of integers we begin with, and regardless of the moves we make.