Belarus Team Selection Test 2019

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– Test 1

1.1 Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(f(n+1))=f(f(n))+2^{n-1}$$

for any positive integer n? (As usual, $\mathbb{N}$ stands for the set of positive integers.)

(I. Gorodnin)

1.2 Points M and N are the midpoints of the sides BC and AD, respectively, of a convex quadrilateral $ABCD$. Is it possible that

$$AB+CD>\max(AM+DM,BN+CN)?$$

(Folklore)

1.3 Given the equation

$$a^b\cdot b^c=c^a$$

in positive integers a, b, and c.
(i) Prove that any prime divisor of a divides b as well.
(ii) Solve the equation under the assumption $b\ge a$.
(iii) Prove that the equation has infinitely many solutions.

(I. Voronovich)

1.4 Let the sequence (a_n) be constructed in the following way:

$$a_1=1,\ a_2=1,\ a_{n+2}=a_{n+1}+\frac{1}{a_n},\ n=1,2,\ldots.$$

Prove that $a_{180}>19$.

(Folklore)

– Test 2

2.1 Given a quadratic trinomial $p(x)$ with integer coefficients such that $p(x)$ is not divisible by 3 for all integers x.
Prove that there exist polynomials $f(x)$ and $h(x)$ with integer coefficients such that

$$p(x) \cdot f(x) + 3h(x) = x^6 + x^4 + x^2 + 1.$$

(I. Gorodnin)

2.2 Let O be the circumcenter and H be the orthocenter of an acute-angled triangle ABC. Point T is the midpoint of the segment AO. The perpendicular bisector of AO intersects the line BC at point S.
Prove that the circumcircle of the triangle AST bisects the segment OH.

(M. Berindeanu, RMC 2018 book)

2.3 1019 stones are placed into two non-empty boxes. Each second Alex chooses a box with an even amount of stones and shifts half of these stones into another box.
Prove that for each k, $1 \leq k \leq 1018$, at some moment there will be a box with exactly k stones.

(O. Izhboldin)

2.4 Cells of 11×11 table are colored with n colors (each cell is colored with exactly one color). For each color, the total amount of the cells of this color is not less than 7 and not greater than 13.
Prove that there exists at least one row or column which contains cells of at least four different colors.

(N. Sedrakyan)

– Test 3

3.1 Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the number of divisors of sn and of sk are equal.

3.2 A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Proposed by Mongolia

3.3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the

stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

– Test 4

4.1 A circle ω with radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

- each triangle from T is inscribed in ω;
- no two triangles from T have a common interior point.

Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

4.2 Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

4.3 Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1,$ and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximum possible value of $a_{2018} - a_{2017}$.

– Test 5

5.1 A function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, satisfies the following condition: for any positive integers m and n ($m > n$) the number $f(m) - f(n)$ is divisible by $m - n$.
Is the function f necessarily a polynomial? (In other words, is it true that for any such function there exists a polynomial $p(x)$ with real coefficients such that $f(n) = p(n)$ for all positive integers n?)

(Folklore)

5.2 Let AA_1 be the bisector of a triangle ABC. Points D and F are chosen on the line BC such that A_1 is the midpoint of the segment DF. A line l, different from BC, passes through A_1 and intersects the lines AB and AC at points B_1 and C_1, respectively.
Find the locus of the points of intersection of the lines B_1D and C_1F for all possible positions of l.

(M. Karpuk)

5.3 A polygon (not necessarily convex) on the coordinate plane is called *plump* if it satisfies the following conditions: • coordinates of vertices are integers; • each side forms an angle of $0°$, $90°$, or $45°$ with the abscissa axis; • internal angles belong to the interval $[90°, 270°]$.
Prove that if a square of each side length of a plump polygon is even, then such a polygon can be cut into several convex plump polygons.

(A. Yuran)

– Test 6

6.1 Two circles Ω and Γ are internally tangent at the point B. The chord AC of Γ is tangent to Ω at the point L, and the segments AB and BC intersect Ω at the points M and N. Let M_1 and N_1 be the reflections of M and N about the line BL; and let M_2 and N_2 be the reflections of M and N about the line AC. The lines M_1M_2 and N_1N_2 intersect at the point K.
Prove that the lines BK and AC are perpendicular.

(M. Karpuk)

6.2 The numbers $1, 2, \dots, 49, 50$ are written on the blackboard. Ann performs the following operation: she chooses three arbitrary numbers a, b, c from the board, replaces them by their sum $a+b+c$ and writes $(a+b)(b+c)(c+a)$ to her notebook. Ann performs such operations until only two numbers remain on the board (in total 24 operations). Then she calculates the sum of all 24 numbers written in the notebook. Let A and B be the maximum and the minimum possible sums that Ann san obtain.
Find the value of $\frac{A}{B}$.

(I. Voronovich)

6.3 Let $n \geq 2018$ be an integer, and let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

– Test 7

7.1 The internal bisectors of angles $\angle DAB$ and $\angle BCD$ of a quadrilateral $ABCD$ intersect at the point X_1, and the external bisectors of these angles intersect at the point X_2. The internal bisectors of angles $\angle ABC$ and $\angle CDA$ intersect at the point Y_1, and the external bisectors of these angles intersect at the point Y_2.
Prove that the angle between the lines X_1X_2 and Y_1Y_2 equals the angle between the diagonals AC and BD.

(A. Voidelevich)

7.2 Define the sequence $a_0, a_1, a_2, \dots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

7.3 Given a positive integer n, determine the maximal constant C_n satisfying the following condition: for any partition of the set $\{1, 2, \dots, 2n\}$ into two n-element subsets A and B, there exist labellings $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ of A and B, respectively, such that

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2 \geq C_n.$$

(B. Serankou, M. Karpuk)

– Test 8

8.1 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

8.2 Let $\mathbb{Z}$ be the set of all integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following conditions:
1. $f(f(x)) = xf(x) - x^2 + 2$ for all $x \in \mathbb{Z}$;
2. f takes all integer values.

(I. Voronovich)

8.3 Prove that for $n > 1$, n does not divide $2^{n-1} + 1$