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– Challenger Division

1 At a math competition, a team of 8 students has 2 hours to solve 30 problems. If each problem needs to be solved by 2 students, on average how many minutes can a student spend on a problem?

2 A *trifecta* is an ordered triple of positive integers (a, b, c) with $a < b < c$ such that a divides b, b divides c, and c divides ab. What is the largest possible sum $a + b + c$ over all trifectas of three-digit integers?

3 Determine all real values of x for which

$$\frac{1}{\sqrt{x}+\sqrt{x-2}}+\frac{1}{\sqrt{x}+\sqrt{x+2}}=\frac{1}{4}.$$

4 How many six-letter words formed from the letters of AMC do not contain the substring AMC? (For example, AMAMMC has this property, but AAMCCC does not.)

5 What is the largest integer with distinct digits such that no two of its digits sum to a perfect square?

6 Seven two-digit integers form a strictly increasing arithmetic sequence. If the first and last terms of this sequence have the same set of digits, what is the sum of all possible medians of the sequence?

7 Triangle ABC has $AB = 8, AC = 12, BC = 10$. Let D be the intersection of the angle bisector of angle A with BC. Let M be the midpoint of BC. The line parallel to AC passing through M intersects AB at N. The line parallel to AB passing through D intersects AC at P. MN and DP intersect at E. Find the area of $ANEP$.

8 The Fibonacci sequence $F_0, F_1, \ldots$ satisfies $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Compute the number of triples (a, b, c) with $0 \leq a < b < c \leq 100$ for which F_a, F_b, F_c is an arithmetic progression.

9 How many decreasing sequences $a_1, a_2, \ldots, a_{2019}$ of positive integers are there such that $a_1 \leq 2019^2$ and $a_n + n$ is even for each $1 \leq n \leq 2019$?

10 Let a, b be positive real numbers with $a > b$. Compute the minimum possible value of the expression

$$\frac{a^2b - ab^2 + 8}{ab - b^2}.$$

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11 Let ABC be a right triangle with hypotenuse AB. Point E is on AB with $AE = 10BE$, and point D is outside triangle ABC such that $DC = DB$ and $\angle CDA = \angle BDE$. Let $[ABC]$ and $[BCD]$ denote the areas of triangles ABC and BCD. Determine the value of $\frac{[BCD]}{[ABC]}$.

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12 Determine the number of 10-letter strings consisting of As, Bs, and Cs such that there is no B between any two As.

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13 The infinite sequence $a_0, a_1, \dots$ is given by $a_1 = \frac{1}{2}$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine the infinite product $a_1 a_2 a_3 \cdots$.

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14 In a circle of radius 10, three congruent chords bound an equilateral triangle with side length 8. The endpoints of these chords form a convex hexagon. Compute the area of this hexagon.

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15 Let $P(x)$ be a polynomial with integer coefficients such that

$$P(\sqrt{2}\sin x) = -P(\sqrt{2}\cos x)$$

for all real numbers x. What is the largest prime that must divide $P(2019)$?

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16 What is the product of the factors of 30^{12} that are congruent to 1 modulo 7?

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17 Tommy takes a 25-question true-false test. He answers each question correctly with independent probability $\frac{1}{2}$. Tommy earns bonus points for correct streaks: the first question in a streak is worth 1 point, the second question is worth 2 points, and so on. For instance, the sequence TFFTTTFT is worth 1 + 1 + 2 + 3 + 1 = 8 points. Compute the expected value of Tommys score.

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18 Two circles with radii 3 and 4 are externally tangent at P. Let $A \neq P$ be on the first circle and $B \neq P$ be on the second circle, and let the tangents at A and B to the respective circles intersect at Q. Given that $QA = QB$ and AB bisects PQ, compute the area of QAB.

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19 Let n be the largest integer such that 5^n divides $12^{2015} + 13^{2015}$. Compute the remainder when $\frac{12^{2015}+13^{2015}}{5^n}$ is divided by 1000.

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20 Kelvin the Frog lives in the 2-D plane. Each day, he picks a uniformly random direction (i.e. a uniformly random bearing $\theta \in [0, 2\pi]$) and jumps a mile in that direction. Let D be the number of miles Kelvin is away from his starting point after ten days. Determine the expected value of D^4.

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21 Let $ABCD$ be a rectangle satisfying $AB = CD = 24$, and let E and G be points on the extension of BA past A and the extension of CD past D respectively such that $AE = 1$ and $DG = 3$.

Suppose that there exists a unique pair of points (F, H) on lines BC and DA respectively such that H is the orthocenter of $\triangle EFG$. Find the sum of all possible values of BC.

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22 Find the largest real number λ such that

$$a_1^2+\cdots+a_{2019}^2 \geq a_1a_2+a_2a_3+\cdots+a_{1008}a_{1009}+\lambda a_{1009}a_{1010}+\lambda a_{1010}a_{1011}+a_{1011}a_{1012}+\cdots+a_{2018}a_{2019}$$

for all real numbers $a_1, \ldots, a_{2019}$. The coefficients on the right-hand side are 1 for all terms except $a_{1009}a_{1010}$ and $a_{1010}a_{1011}$, which have coefficient λ.

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23 For Kelvin the Frog's birthday, Alex the Kat gives him one brick weighing x pounds, two bricks weighing y pounds, and three bricks weighing z pounds, where x, y, z are positive integers of Kelvin the Frog's choice.

Kelvin the Frog has a balance scale. By placing some combination of bricks on the scale (possibly on both sides), he wants to be able to balance any item of weight $1, 2, \ldots, N$ pounds. What is the largest N for which Kelvin the Frog can succeed?

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24 Let ABC be a triangle with $\angle A = 60^\circ$, $AB = 12$, $AC = 14$. Point D is on BC such that $\angle BAD = \angle CAD$. Extend AD to meet the circumcircle at M. The circumcircle of BDM intersects AB at $K \neq B$, and line KM intersects the circumcircle of CDM at $L \neq M$. Find $\frac{KM}{LM}$.

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25 Determine the remainder when

$$\prod_{i=1}^{2016}(i^4+5)$$

is divided by 2017.

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26 The permutations of $OLYMPIAD$ are arranged in lexicographical order, with $ADILMOPY$ being arrangement 1 and its reverse being arrangement 40320. Yu Semo and Yu Sejmo both choose a uniformly random arrangement. The immature Yu Sejmo exclaims, "My fourth letter is L!" while Yu Semo remains silent. Given this information, let E_1 be the expected arrangement number of Yu Semo and E_2 be the expected arrangement number of Yu Sejmo. Compute $E_2 - E_1$.

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27 For an integer n, define $f(n)$ to be the greatest integer k such that 2^k divides $\binom{n}{m}$ for some $0 \leq m \leq n$. Compute $f(1) + f(2) + \cdots + f(2048)$.

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28 Alex the Kat plays the following game. First, he writes the number 27000 on a blackboard. Each minute, he erases the number on the blackboard and replaces it with a number chosen uniformly randomly from its positive divisors, including itself. Find the probability that, after 2019 minutes, the number on the blackboard is 1.

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29 Let n be a positive integer, and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers. Alex the Kat writes down the n^2 numbers of the form $\min(a_i, a_j)$, and Kelvin the Frog writes down the n^2 numbers of the form $\max(b_i, b_j)$.

Let x_n be the largest possible size of the set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, such that Alex the Kat and Kelvin the Frog write down the same collection of numbers. Determine the number of distinct integers in the sequence $x_1, x_2, \ldots, x_{10,000}$.

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30 Let ABC be a triangle with $BC = a$, $CA = b$, and $AB = c$. The A-excircle is tangent to $\overline{BC}$ at A_1; points B_1 and C_1 are similarly defined.

Determine the number of ways to select positive integers a, b, c such that

- the numbers $-a + b + c$, $a - b + c$, and $a + b - c$ are even integers at most 100, and
- the circle through the midpoints of $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ is tangent to the incircle of $\triangle ABC$.

– Premier Division

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1 Kelvin the Frog and Alex the Kat are playing a game on an initially empty blackboard. Kelvin begins by writing a digit. Then, the players alternate inserting a digit anywhere into the number currently on the blackboard, including possibly a leading zero (e.g. 12 can become 123, 142, 512, 012, etc.). Alex wins if the blackboard shows a perfect square at any time, and Kelvin's goal is prevent Alex from winning. Does Alex have a winning strategy?

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2 Let $n \geq 2$ be an even integer. Find the maximum integer k (in terms of n) such that 2^k divides $\binom{n}{m}$ for some $0 \leq m \leq n$.

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3 Let ABC be a scalene triangle. The incircle of ABC touches $\overline{BC}$ at D. Let P be a point on $\overline{BC}$ satisfying $\angle BAP = \angle CAP$, and M be the midpoint of $\overline{BC}$. Define Q to be on $\overline{AM}$ such that $\overline{PQ} \perp \overline{AM}$. Prove that the circumcircle of $\triangle AQD$ is tangent to $\overline{BC}$.

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4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x) + y)^2 = (x - y)(f(x) - f(y)) + 4f(x)f(y).$$

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5 The number 2019 is written on a blackboard. Every minute, if the number a is written on the board, Evan erases it and replaces it with a number chosen from the set

$$\{0, 1, 2, \dots, \lceil 2.01a \rceil\}$$

uniformly at random. Is there an integer N such that the board reads 0 after N steps with at least 99% probability?

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6 A *mirrored polynomial* is a polynomial f of degree 100 with real coefficients such that the x^{50} coefficient of f is 1, and $f(x) = x^{100} f(1/x)$ holds for all real nonzero x. Find the smallest real constant C such that any mirrored polynomial f satisfying $f(1) \geq C$ has a complex root z obeying $|z| = 1$.

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7 Let $AXBY$ be a convex quadrilateral. The incircle of $\triangle AXY$ has center I_A and touches $\overline{AX}$ and $\overline{AY}$ at A_1 and A_2 respectively. The incircle of $\triangle BXY$ has center I_B and touches $\overline{BX}$ and $\overline{BY}$ at B_1 and B_2 respectively. Define $P = \overline{XI_A} \cap \overline{YI_B}$, $Q = \overline{XI_B} \cap \overline{YI_A}$, and $R = \overline{A_1B_1} \cap \overline{A_2B_2}$.

- Prove that if $\angle AXB = \angle AYB$, then P, Q, R are collinear.
- Prove that if there exists a circle tangent to all four sides of $AXBY$, then P, Q, R are collinear.

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8 Find all pairs of positive integers (m, n) such that $(2^m - 1)(2^n - 1)$ is a perfect square.
