Mexico National Olympiad 2018

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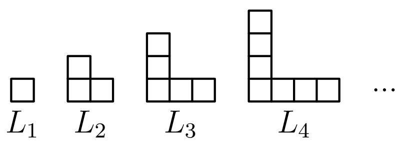
by parmenides51, juckter, plagueis

– Day 1

1 Let A and B be two points on a line ℓ, M the midpoint of AB, and X a point on segment AB other than M. Let Ω be a semicircle with diameter AB. Consider a point P on Ω and let Γ be the circle through P and X that is tangent to AB. Let Q be the second intersection point of Ω and Γ. The internal angle bisector of $\angle PXQ$ intersects Γ at a point R. Let Y be a point on ℓ such that RY is perpendicular to ℓ. Show that $MX > XY$

2 For each positive integer m, we define L_m as the figure that is obtained by overlapping two $1 \times m$ and $m \times 1$ rectangles in such a way that they coincide at the 1×1 square at their ends, as shown in the figure.



Using some figures $L_{m_1}, L_{m_2}, \dots, L_{m_k}$, we cover an $n \times n$ board completely, in such a way that the edges of the figure coincide with lines in the board. Among all possible coverings of the board, find the minimal possible value of $m_1 + m_2 + \dots + m_k$.

Note: In covering the board, the figures may be rotated or reflected, and they may overlap or not be completely contained within the board.

3 A sequence $a_2, a_3, \dots, a_n$ of positive integers is said to be *campechana*, if for each i such that $2 \le i \le n$ it holds that exactly a_i terms of the sequence are relatively prime to i. We say that the *size* of such a sequence is $n - 1$. Let $m = p_1 p_2 \dots p_k$, where $p_1, p_2, \dots, p_k$ are pairwise distinct primes and $k \geq 2$. Show that there exist at least two different campechana sequences of size m.

– Day 2

4 Let $n \geq 2$ be an integer. For each k-tuple of positive integers $a_1, a_2, \dots, a_k$ such that $a_1 + a_2 + \dots + a_k = n$, consider the sums $S_i = 1 + 2 + \dots + a_i$ for $1 \le i \le k$. Determine, in terms of n, the maximum possible value of the product $S_1 S_2 \cdots S_k$.

Proposed by Misael Pelayo

5 Let $n \geq 5$ an integer and consider a regular n-gon. Initially, Nacho is situated in one of the vertices of the n-gon, in which he puts a flag. He will start moving clockwise. First, he moves one position and puts another flag, then, two positions and puts another flag, etcetera, until he finally moves $n-1$ positions and puts a flag, in such a way that he puts n flags in total. ¿For which values of n, Nacho will have put a flag in each of the n vertices?

6 Let ABC be an acute-angled triangle with circumference Ω. Let the angle bisectors of $\angle B$ and $\angle C$ intersect Ω again at M and N. Let I be the intersection point of these angle bisectors. Let M' and N' be the respective reflections of M and N in AC and AB. Prove that the center of the circle passing through I, M', N' lies on the altitude of triangle ABC from A.

Proposed by Victor Domínguez and Ariel García