

Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2011

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by parmenides51

- 1** Let n be the number that is produced by concatenating the numbers $1, 2, \dots, 4022$, that is, $n = 1234567891011\dots40214022$.
- Show that n is divisible by 3.
 - Let $a_1 = n^{2011}$, and let a_i be the sum of the digits of a_{i-1} for $i > 1$. Find a_4 .

- 2a** In the quadrilateral $ABCD$ the side AB has length 7, BC length 14, CD length 26, and DA length 23. Show that the diagonals are perpendicular.

You may assume that the quadrilateral is convex (all internal angles are less than 180°).

- 2b** The diagonals $AD, BE,$ and CF of a convex hexagon $ABCDEF$ intersect in a common point. Show that $a(ABE)a(CDA)a(EFC) = a(BCE)a(DEA)a(FAC)$, where $a(KLM)$ is the area of the triangle KLM .

<https://cdn.artofproblemsolving.com/attachments/0/a/bcbbddedde159150fe3c26b1f0a2bfc322aa1.png>

- 3a** The positive numbers a_1, a_2, \dots satisfy $a_1 = 1$ and $(m+n)a_{m+n} \leq a_m + a_n$ for all positive integers m and n . Show that $\frac{1}{a_{200}} > 4 \cdot 10^7$.

- 3b** Find all functions f from the real numbers to the real numbers such that $f(xy) \leq \frac{1}{2}(f(x) + f(y))$ for all real numbers x and y .

- 4a** In a town there are n avenues running from south to north. They are numbered 1 through n (from west to east). There are n streets running from west to east they are also numbered 1 through n (from south to north). If you drive through the junction of the k th avenue and the ℓ th street, you have to pay $k\ell$ kroner. How much do you at least have to pay for driving from the junction of the 1st avenue and the 1st street to the junction of the n th avenue and the n th street? (You also pay for the starting and finishing junctions.)

- 4b** In a group of 199 persons, each person is a friend of exactly 100 other persons in the group. All friendships are mutual, and we do not count a person as a friend of himself/herself. For which integers $k > 1$ is the existence of k persons, all being friends of each other, guaranteed?