Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2007

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by parmenides51

1 We consider the sum of the digits of a positive integer.
For example, the sum of the digits of 2007 is equal to 9, since $2+0+0+7=9$.
(a) How many integers n, where $0<n<100000$, have an even sum of digits?
(b) How many integers n, where $0<n<100000$, have a sum of digits that is less than or equal to 22?

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2 The vertices of a convex pentagon $ABCDE$ lie on a circle γ_1.
The diagonals AC, CE, EB, BD, and DA are tangents to another circle γ_2 with the same centre as γ_1.
(a) Show that all angles of the pentagon $ABCDE$ have the same size and that all edges of the pentagon have the same length.
(b) What is the ratio of the radii of the circles γ_1 and γ_2? (The answer should be given in terms of integers, the four basic arithmetic operations and extraction of roots only.)

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3 (a) Let x and y be two positive integers such that $\sqrt{x}+\sqrt{y}$ is an integer.
Show that $\sqrt{x}$ and $\sqrt{y}$ are both integers.
(b) Find all positive integers x and y such that $\sqrt{x}+\sqrt{y}=\sqrt{2007}$.

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4 Let a, b and c be integers such that $a+b+c=0$.
(a) Show that $a^4+b^4+c^4$ is divisible by $a^2+b^2+c^2$.
(b) Show that $a^{100}+b^{100}+c^{100}$ is divisible by $a^2+b^2+c^2$.
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