**Federal Competition For Advanced Students, Part 2 2010**

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by parmenides51

– Day 1

**1** Show that $\frac{(x-y)^7+(y-z)^7+(z-x)^7-(x-y)(y-z)(z-x)((x-y)^4+(y-z)^4+(z-x)^4)}{(x-y)^5+(y-z)^5+(z-x)^5} \geq 3$
holds for all triples of distinct integers $x, y, z$. When does equality hold?

**2** Determine all triples $(x, y, z)$ of positive integers $x > y > z > 0$, such that $x^2 = y \cdot 2^z + 1$

**3** On a circular billiard table a ball rebounds from the rails as if the rail was the tangent to the circle at the point of impact.
A regular hexagon with its vertices on the circle is drawn on a circular billiard table.
A (point-shaped) ball is placed somewhere on the circumference of the hexagon, but not on one of its edges.
Describe a periodical track of this ball with exactly four points at the rails.
With how many different directions of impact can the ball be brought onto such a track?

– Day 2

**4** Consider the part of a lattice given by the corners $(0, 0)$, $(n, 0)$, $(n, 2)$ and $(0, 2)$.
From a lattice point $(a, b)$ one can move to $(a + 1, b)$ or to $(a + 1, b + 1)$ or to $(a, b - 1)$, provided that the second point is also contained in the part of the lattice.
How many ways are there to move from $(0, 0)$ to $(n, 2)$ considering these rules?

**5** Two decompositions of a square into three rectangles are called substantially different, if re-ordering the rectangles does not change one into the other.
How many substantially different decompositions of a $2010 \times 2010$ square in three rectangles with integer side lengths are there such that the area of one rectangle is equal to the arithmetic mean of the areas of the other rectangles?

**6** A diagonal of a convex hexagon is called *long* if it decomposes the hexagon into two quadrangles.
Each pair of *long* diagonals decomposes the hexagon into two triangles and two quadrangles.
Given is a hexagon with the property, that for each decomposition by two *long* diagonals the resulting triangles are both isosceles with the side of the hexagon as base.
Show that the hexagon has a circumcircle.