International Olympiad of Metropolises 2019

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by parmenides51, HKIS200543

Day 1 September 3rd, 2019

1 Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p+n}{qr}, \frac{q+n}{rp}, \frac{r+n}{pq}$$

are integers. Prove that $p = q = r$.

Nazar Agakhanov

2 In a social network with a fixed finite setback of users, each user had a fixed set of *followers* among the other users. Each user has an initial positive integer rating (not necessarily the same for all users). Every midnight, the rating of every user increases by the sum of the ratings that his followers had just before midnight.

Let m be a positive integer. A hacker, who is not a user of the social network, wants all the users to have ratings divisible by m. Every day, he can either choose a user and increase his rating by 1, or do nothing. Prove that the hacker can achieve his goal after some number of days.

Vladislav Novikov

3 In a non-equilateral triangle ABC point I is the incenter and point O is the circumcenter. A line s through I is perpendicular to IO. Line ℓ symmetric to like BC with respect to s meets the segments AB and AC at points K and L, respectively (K and L are different from A). Prove that the circumcenter of triangle AKL lies on the line IO.

Duan Djuki

Day 2 September 4th, 2019

4 There are 100 students taking an exam. The professor calls them one by one and asks each student a single person question: How many of 100 students will have a passed mark by the end of this exam? The students answer must be an integer. Upon receiving the answer, the professor immediately publicly announces the students mark which is either passed or failed.

After all the students have got their marks, an inspector comes and checks if there is any student who gave the correct answer but got a failed mark. If at least one such student exists, then the professor is suspended and all the marks are replaced with passed. Otherwise no changes are made.

Can the students come up with a strategy that guarantees a passed mark to each of them?

Denis Afrizonov

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5 We are given a convex four-sided pyramid with apex S and base face $ABCD$ such that the pyramid has an inscribed sphere (i.e., it contains a sphere which is tangent to each race). By making cuts along the edges SA, SB, SC, SD and rotating the faces SAB, SBC, SCD, SDA outwards into the plane $ABCD$, we unfold the pyramid into the polygon $AKBLCMDN$ as shown in the figure. Prove that K, L, M, N are concyclic.

Tibor Bakos and Gza Ks

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6 Let p be a prime and let $f(x)$ be a polynomial of degree d with integer coefficients. Assume that the numbers $f(1), f(2), \dots, f(p)$ leave exactly k distinct remainders when divided by p, and $1 < k < p$. Prove that

$$\frac{p-1}{d} \leq k-1 \leq (p-1)\left(1-\frac{1}{d}\right).$$

Dniel Domn, Gauls Krolyi, and Emil Kiss
