Dutch Mathematical Olympiad 2009

www.artofproblemsolving.com/community/c945636

by parmenides51

1 In this problem, we consider integers consisting of 5 digits, of which the rst and last one are nonzero. We say that such an integer is a palindromic product if it satis es the following two conditions:
- the integer is a palindrome, (i.e. it doesn't matter if you read it from left to right, or the other way around);
- the integer is a product of two positive integers, of which the fi rst, when read from left to right, is equal to the second, when read from right to left, like 4831 and 1384.
For example, 20502 is a palindromic product, since $102 \cdot 201 = 20502$, and 20502 itself is a palindrome.
Determine all palindromic products of 5 digits.

---

2 Consider the sequence of integers $0, 1, 2, 4, 6, 9, 12, ...$ obtained by starting with zero, adding 1, then adding 1 again, then adding 2, and adding 2 again, then adding 3, and adding 3 again, and so on. If we call the subsequent terms of this sequence $a_0, a_1, a_2, ...$, then we have $a_0 = 0$, and $a_{2n-1} = a_{2n-2} + n$, $a_{2n} = a_{2n-1} + n$ for all integers $n \geq 1$.
Find all integers $k \geq 0$ for which a_k is the square of an integer.

---

3 A tennis tournament has at least three participants. Every participant plays exactly one match against every other participant. Moreover, every participant wins at least one of the matches he plays. (Draws do not occur in tennis matches.)
Show that there are three participants A, B and C for which the following holds: A wins against B, B wins against C, and C wins against A.

---

4 Let ABC be an arbitrary triangle. On the perpendicular bisector of AB, there is a point P inside of triangle ABC. On the sides BC and CA, triangles BQC and CRA are placed externally. These triangles satisfy $\triangle BPA \sim \triangle BQC \sim \triangle CRA$. (So Q and A lie on opposite sides of BC, and R and B lie on opposite sides of AC.) Show that the points P, Q, C and R form a parallelogram.

---

5 We number a hundred blank cards on both sides with the numbers 1 to 100. The cards are then stacked in order, with the card with the number 1 on top.
The order of the cards is changed step by step as follows: at the 1st step the top card is turned around, and is put back on top of the stack (nothing changes, of course), at the 2nd step the topmost 2 cards are turned around, and put back on top of the stack, up to the 100th step, in which the entire stack of 100 cards is turned around. At the 101st step, again only the top card is turned around, at the 102nd step, the top most 2 cards are turned around, and so on.
Show that after a finite number of steps, the cards return to their original positions.
