Dutch Mathematical Olympiad 2012

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1 Let a, b, c, and d be four distinct integers.
Prove that $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.

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2 We number the columns of an $n \times n$-board from 1 to n. In each cell, we place a number. This is done in such a way that each row precisely contains the numbers 1 to n (in some order), and also each column contains the numbers 1 to n (in some order). Next, each cell that contains a number greater than the cell's column number, is coloured grey. In the figure below you can see an example for the case $n = 3$.

1	2	3
3	1	2
1	2	3
2	3	1

(a) Suppose that $n = 5$. Can the numbers be placed in such a way that each row contains the same number of grey cells?
(b) Suppose that $n = 10$. Can the numbers be placed in such a way that each row contains the same number of grey cells?

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3 Determine all pairs (p, m) consisting of a prime number p and a positive integer m, for which $p^3 + m(p+2) = m^2 + p + 1$ holds.

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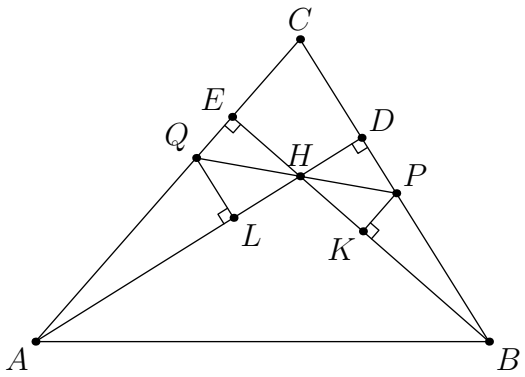
4 We are given an acute triangle ABC and points D on BC and E on AC such that AD is perpendicular to BC and BE is perpendicular to AC. The intersection of AD and BE is called H. A line through H intersects line segment BC in P, and intersects line segment AC in Q. Furthermore, K is a point on BE such that PK is perpendicular to BE, and L is a point on AD such that QL is perpendicular to AD. Prove that DK and EL are parallel.



5 The numbers 1 to 12 are arranged in a sequence. The number of ways this can be done equals $12 \times 11 \times 10 \times ... \times 1$. We impose the condition that in the sequence there should be exactly one number that is smaller than the number directly preceding it.
How many of the $12 \times 11 \times 10 \times ... \times 1$ sequences satisfy this condition?