Dutch Mathematical Olympiad 2016

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by parmenides51

1 (a) On a long pavement, a sequence of 999 integers is written in chalk. The numbers need not be in increasing order and need not be distinct. Merlijn encircles 500 of the numbers with red chalk. From left to right, the numbers circled in red are precisely the numbers $1, 2, 3, ..., 499, 500$. Next, Jeroen encircles 500 of the numbers with blue chalk. From left to right, the numbers circled in blue are precisely the numbers $500, 499, 498, ..., 2, 1$.

Prove that the middle number in the sequence of 999 numbers is circled both in red and in blue.

(b) Merlijn and Jeroen cross the street and find another sequence of 999 integers on the pavement. Again Merlijn circles 500 of the numbers with red chalk. Again the numbers circled in red are precisely the numbers $1, 2, 3, ..., 499, 500$ from left to right. Now Jeroen circles 500 of the numbers, not necessarily the same as Merlijn, with green chalk. The numbers circled in green are also precisely the numbers $1, 2, 3, ..., 499, 500$ from left to right.

Prove: there is a number that is circled both in red and in green that is not the middle number of the sequence of 999 numbers.

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2 For an integer $n \ge 1$ we consider sequences of $2n$ numbers, each equal to $0, -1$ or 1. The *sum product value* of such a sequence is calculated by first multiplying each pair of numbers from the sequence, and then adding all the results together.
For example, if we take $n = 2$ and the sequence $0, 1, 1, -1$, then we find the products $0 \cdot 1, 0 \cdot 1, 0 \cdot -1, 1 \cdot 1, 1 \cdot -1, 1 \cdot -1$. Adding these six results gives the sum product value of this sequence: $0 + 0 + 0 + 1 + (-1) + (-1) = -1$. The sum product value of this sequence is therefore smaller than the sum product value of the sequence $0, 0, 0, 0$, which equals 0.
Determine for each integer $n \ge 1$ the smallest sum product value that such a sequence of $2n$ numbers could have.

Attention: you are required to prove that a smaller sum product value is impossible.

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3 Find all possible triples (a, b, c) of positive integers with the following properties:
• $gcd(a, b) = gcd(a, c) = gcd(b, c) = 1$,
• a is a divisor of $a + b + c$,
• b is a divisor of $a + b + c$,
• c is a divisor of $a + b + c$.

(Here $gcd(x, y)$ is the greatest common divisor of x and y.)

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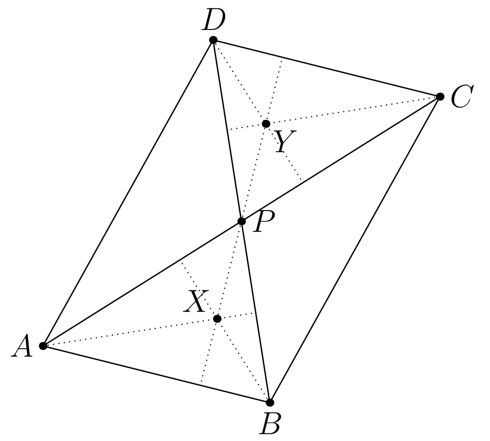
4 juniors In a quadrilateral $ABCD$ the intersection of the diagonals is called P. Point X is the orthocentre of triangle PAB. (The orthocentre of a triangle is the point where the three altitudes of the triangle intersect.) Point Y is the orthocentre of triangle PCD. Suppose that X lies inside triangle PAB and Y lies inside triangle PCD. Moreover, suppose that P is the midpoint of line segment XY . Prove that $ABCD$ is a parallelogram.



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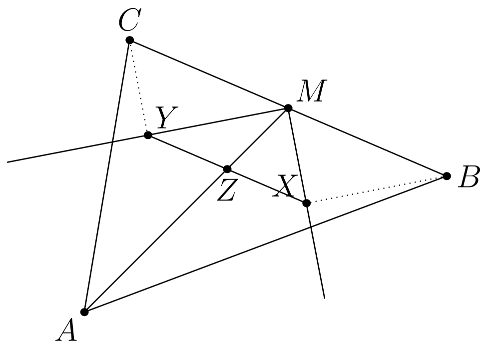
4 seniors In the acute triangle ABC, the midpoint of side BC is called M. Point X lies on the angle bisector of $\angle AMB$ such that $\angle BXM = 90^o$. Point Y lies on the angle bisector of $\angle AMC$ such that $\angle CYM = 90^o$. Line segments AM and XY intersect in point Z. Prove that Z is the midpoint of XY .



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5 Bas has coloured each of the positive integers. He had several colours at his disposal. His colouring satis es the following requirements:
• each odd integer is coloured blue,
• each integer n has the same colour as $4n$,
• each integer n has the same colour as at least one of the integers $n+2$ and $n+4$.
Prove that Bas has coloured all integers blue.
