Finnish National High School Mathematics Competition 2019

www.artofproblemsolving.com/community/c947605

by parmenides51

1 Solve $x(8\sqrt{1-x}+\sqrt{1+x}) \le 11\sqrt{1+x}-16\sqrt{1-x}$ when $0 < x \le 1$

---

2 Prove that the number $\lfloor (2+\sqrt{5})^{2019} \rfloor$ is not prime.

---

3 Let $ABCD$ be a cyclic quadrilateral whose side AB is at the same time the diameter of the circle. The lines AC and BD intersect at point E and the extensions of lines AD and BC intersect at point F. Segment EF intersects the circle at G and the extension of segment EF intersects AB at H. Show that if G is the midpoint of FH, then E is the midpoint of GH.

---

4 Define a sequence $a_n = n^n + (n-1)^{n+1}$ when n is a positive integer.
Define all those positive integer m , for which this sequence of numbers is eventually periodic modulo m, e.g. there are such positive integers K and s such that $a_k \equiv a_{k+s}\, (mod\, m)$, where k is an integer with $k \ge K$.

---

5 A teacher is known to have 2^k apples for some $k \in \mathbb{N}$. He ets one of the apples and distributes the rest of the apples to his students A and B. The students do not see how many apples the other gets, and they do not know the number k. However, they have pre-selected a discreet way to reveal one another something about the number of apples: each of the students scratches their head either by their right, left or both hands, depending on the number of apples they have received. To the teacher's surprise, the students will always know which one of the students got more apples, or that the teacher ate the only apple by herself. How is this possible?
