China Second Round Olympiad 2019

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– Test 2

– (A)

1 In acute triangle $\triangle ABC$, M is the midpoint of segment BC. Point P lies in the interior of $\triangle ABC$ such that AP bisects $\angle BAC$. Line MP intersects the circumcircles of $\triangle ABP, \triangle ACP$ at D, E respectively. Prove that if $DE = MP$, then $BC = 2BP$.

2 Let $a_1, a_2, \cdots, a_n$ be integers such that $1 = a_1 \le a_2 \le \cdots \le a_{2019} = 99$. Find the minimum f_0 of the expression

$$f = (a_1^2 + a_2^2 + \cdots + a_{2019}^2) - (a_1a_3 + a_2a_4 + \cdots + a_{2017}a_{2019}),$$

and determine the number of sequences $(a_1, a_2, \cdots, a_n)$ such that $f = f_0$.

3 Let m be an integer where $|m| \ge 2$. Let $a_1, a_2, \cdots$ be a sequence of integers such that a_1, a_2 are not both zero, and for any positive integer n, $a_{n+2} = a_{n+1} - ma_n$.

Prove that if positive integers $r > s \ge 2$ satisfy $a_r = a_s = a_1$, then $r - s \ge |m|$.

4 Let V be a set of 2019 points in space where any of the four points are not on the same plane, and E be the set of edges connected between them. Find the smallest positive integer n satisfying the following condition: if E has at least n elements, then there exists 908 two-element subsets of E such that
-The two edges in each subset share a common vertice,
-Any of the two subsets do not intersect.

– (B)

1 Suppose that $a_1, a_2, \cdots, a_{100} \in \mathbb{R}^+$ such that $a_i \ge a_{101-i}$ $(i = 1, 2, \cdots, 50)$.
Let $x_k = \frac{ka_{k+1}}{a_1+a_2+\cdots+a_k}$ $(k = 1, 2, \cdots, 99)$. Prove that

$$x_1x_2^2\cdots x_{99}^{99} \le 1.$$

2 Find all the positive integers n such that: (1) n has at least 4 positive divisors. (2) if all positive divisors of n are $d_1, d_2, \cdots, d_k$, then $d_2 - d_1, d_3 - d_2, \cdots, d_k - d_{k-1}$ form a geometric sequence.

3 Point A, B, C, D, E lie on a line in this order, such that $BC = CD = \sqrt{AB \cdot DE}$, P doesn't lie on the line, and satisfys that $PB = PD$. Point K, L lie on the segment PB, PD, respectively, such that KC bisects $\angle BKE$, and LC bisects $\angle ALD$.
Prove that A, K, L, E are concyclic.

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4 Each side of a convex 2019-gon polygon is dyed with red, yellow and blue, and there are exactly 673 sides of each kind of color. Prove that there exists at least one way to draw 2016 diagonals to divide the convex 2019-gon polygon into 2017 triangles, such that any two of the 2016 diagonals don't have intersection inside the 2019-gon polygon,and for any triangle in all the 2017 triangles, the colors of the three sides of the triangle are all the same, either totally different.
